INTRODUCTION
to
EXTENDED ELECTRODYNAMICS

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0.1 Foreword

Classical Electrodynamics is probably the most fascinating and complete part of the Classical Field Theory. Intuition, free thought, perspicuity and research skill of many years finally brought about the synthesis of experiment and theory, of physics and mathematics, which we have been calling for short the *Maxwell equations* for the duration of a century and a half. From the beginning of the second half of the 19th century till its end these equations turned from abstract theory into daily practice, as they are today. Their profound study during the first half of the 20th century brought forward a new theoretical concept in physics known as *relativism*. Brave and unprejudiced workers enriched and widened the synthesis achieved through the Maxwell equations, and created a new synthesis known briefly as *quantum electrodynamics*. Every significant scientific breakthrough is based on two things: *respect for the workers and their work and respect for the truth.* "May everyone be respected as a personality, and nobody as an idol" one of the old workers used to say. We may paraphrase that saying: "may every scientific truth be respected, but no one be turned into dogma".

In this book I tried to follow the values this creed teaches, as far as my humble abilities allow me to. Together with the analysis of the classical electrodynamics and the quantum concept of the structure of the electromagnetic field, the path followed brought me to the conclusion that a *soliton-like solution of appropriate non-linear equations characterized by an intrinsic periodical process is the most adequate mathematical model of the basic structural unit of the field - the photon.* The fact that neither the Maxwell equations nor the quantum electrodynamics offer the appropriate tools to find such solutions, unmistakably emphasizes the necessity to look for new equations. The leading physical ideas in this search were the dual ("electro-magnetic") nature of the field on the one hand and the local Energy-Momentum Conservation Laws on the other. The realization that every such soliton-like solution determines in an invariant way its own *scale factor*, as well as the suitable interpretation of the famous formula of Planck for the relation between the full energy E of the photon and the frequency $\nu = 1/T$ of the beforementioned periodical process, which I prefer to write down as $h = E.T$, advanced Planck's constant h to the rank of an estimator, separating the realistic soliton-like models of the photon from among the rest. The resulting soliton-like solutions possess all integral qualities of the photon, as described by quantum electrodynamics, but also a structure, organically tied to an in-

trinsic periodical process, which in its turn generates an intrinsic mechanical momentum - spin. I consider this soliton- like oscillating non-linear wave much more clear and understandable than the *ambiguous " particle-wave " duality.*

The dual 2-component nature of the field predetermined to a great extent the generalization of the equations in the case of an interaction with another continuous physical object, briefly called medium. The proposed physical interpretation of the classical Frobenius equations for complete integrability of a system of non-linear partial differential equations as a *criterium for the absence of dissipation*, turned out to be relevant and was effectively used. The fruitfulness of the new non-linear equations is clearly shown by the family of solutions, giving (3+1)-dimensional interpretation of all (1+1)-dimensional 1-soliton and multisoliton solutions. I have to say that the use of differential geometry proved extremely useful.

This book is addressed to all who love theoretical physics and try to build up their own point of view while in the same time show due respect for others' opinions. The stress is laid on the conceptual and generic framework, while in many cases the intermediate calculations were omitted. I did not propose examples of complete description of actual systems, as this was not my purpose. A feature of this book is the lack of citations. In my opinion the reference list in the end will be enough for that purpose.

I would like to express my most sincere gratitude to all my friends and colleagues, with whom I discussed to one extent or another the issues and results herein presented. Each and every conversation was extremely valuable to me both as an actual analysis of the issue and as a stimulus for its further study. I would highly appreciate any remarks and opinions concerning this book from anyone interested in the subject matter being analyzed.

0.2 To the Reader

Dear Reader,

Opening the first pages of a new book, devoted to the part of science you like and want to know as better as you can, the first and most natural question you ask is: what new shall I learn if I read this book? Having in view the huge quantity of monographs and other issues, devoted to every of the various and numerous branches of physics, the natural respect to you requires a short but sufficiently informative and fair response to this question. And this is the contents of these preliminary notes. In other words, I'll try to explain in short what is this *extended electrodynamics* and what aims to achieve this extension of the well known from the University electrodynamics of Faraday and Maxwell.

First of all, let me specify the kind of extension I mean. In theoretical physics by means of mathematical relations and equations some of the really existing objects and processes are modelled and described. An equation separates those values of a given quantity, which are of some interest to that person, who has written this equation. For example, the elementary equation $x-1=0$ separates the value 1 of the variable x. If, for some reasons, we want to separate two values of the variable x, e.g. $(1,-1)$, we, naturally, write down a new equation, which has two solutions: 1 and (-1). The new equation will be $x^2-1=(x+1)(x-1)=0$. In this way we extended the set of admissible values of the variable x, or we may say that we have extended the equation $x-1=0$ to the equation $x^2-1=0$. The idea is now quite clear: *we write down new (algebraic or differential) equation, whose solutions comprise as a subclass all solutions of the old equation, but have also new ones*. The important point is the new solutions to have the properties, desired by us.

The basic equations of Classical electrodynamics are the well known equations of Faraday-Maxwell. So when we talk about *extended electrodynamics* it is clear that just these equations we have to extend in the sense, mentioned above. In order to motivate such an extension we should honestly say two things: which properties of Maxwell equations we do not like sufficiently, and what new properties will be required from the new solutions of the extended equations. Let me briefly discuss this matter.

As it is well known, at zero electric current Maxwell's equations require the components of the electric and magnetic vectors to satisfy the wave equation (d'Alembert's equation). The solutions of this equation in the whole 3-

space have the following 2 properties: first, every solution can be represented by a sum (finite or infinite) of a subclass of solutions, known as *plane waves*; second, every solution, defined by some finite localized and smooth enough initial condition, "blows up" radially and goes to infinity with the velocity of light. The plane waves are characterized by the condition, that there exists a system of canonical coordinates on $\mathcal{R}^3$ in which these solutions depend on one space variable only. This means, that at every moment every such solution occupies the whole 3-space, or an infinite its subregion. Therefore, they are *infinite* and their integral energy is infinitely large. We note that this infinity does not come from some singularity of the solutions. We set the question: do there exist in Nature electromagnetic fields with properties adequate to these *exact* solutions called 'plane waves'? The natural answer is negative, since in order to create such an object it will be necessary to transform an infinite quantity of some kind of energy into electromagnetic energy, which will take infinite period of time.

On the other hand, according to the second property, every finite solution is strongly time-unstable, so it could hardly be considered as a realistic model of a real object. As for the static case, the components of the electric and magnetic vectors satisfy the Laplace equation and, as it is well known the solutions of this equation in the whole 3-space are singular, or when they are finite and nonsingular, they are constant.We conclude, that the vacuum Maxwell equations can not describe finite and time-stable, i.e. *soliton-like* electromagnetic formations. And *our purpose is to describe just such soliton-like electromagnetic formations*.

If we now recall the basic idealization of classical mechanics, the material point, we'll see that it is a full antipode of the plane wave: the material point *has no structure, occupies zero volume* and has finite energy. Are there such objects in Nature? According to my opinion the real objects are finite, i.e. at every moment of its existence every really existing object occupies finite, comparatively small volume and it has definite properties of constancy and stability. When they are subject to external perturbations they survive or transform into other objects, obeying definite conservation laws. Usually, a survival is connected with some change in the behavior of the object as a whole, but it also keeps some of its basic characteristics unchanged, otherwise, we could not say that the object is the same, i.e its identification after the perturbation should be possible.

At the beginning of our century it has become clear that the electromagnetic field has a discrete nature, i.e. it consists of many non-interacting

(or very weakly interacting) objects, carrying energy, momentum and intrinsic angular momentum. Moreover, *the integral energy of these objects originates intrinsically from a periodic process with frequency of* ν in correspondence with the Planck's formula $E = h\nu$. The experiment shows soon that these objects are finite, they move as a whole with the velocity of light along straight lines, carry momentum $h\nu/c$ and intrinsic angular momentum $h = ET = E/\nu$. The problem to describe such (free) objects, called later photons, appeared. Because of the finite nature and time-stability of photons it is clear that the solutions of the wave equation can not give good enough mathematical models of them. On the other hand, the material point (or particle) can not also serve as a model, since there is no any sensible way to assign the characteristic *frequency* to a free particle. The frequency always comes from an outside elastic force, so it depends strongly on this force and is not a proper characteristic of the particle. The classical free particle moves always along straight lines with a constant velocity. There have been some unsuccessful efforts to build an extended model of the photon, but pressed by the quickly advancing experiment, the major part of the physicists assume the prescription schemes of quantum electrodynamics. A basic assumption in these schemes is that the photon is a point-like object, so its frequency (or spin) is an integral characteristic of the same kind as the proper mass and the electric charge. I do not share this point of view, since *a free structureless object can not carry the physical characteristic of intrinsic angular momentum and no periodic process can be associated with its existence*. I think that *a periodic process and an intrinsic angular momentum can be associated with a free object only if it has structure*, and these characteristics may have finite values only if the object is finite. The photon moves uniformly as a whole, but it is not a point-like (or structureless) object, and its existence is strongly connected with some intrinsic periodic process with a constant frequency. This process occurs in the whole volume and in this way generates the intrinsic angular momentum. Since at this stage we do not see any other more-natural way to combine the known photon's features than the notion of soliton-like objects, we turn to appropriate (3+1)-soliton-like solutions of a definite system of nonlinear partial differential equations as possible mathematical models.

As it was mentioned earlier, Maxwell's equations have no such solutions. On the other hand, as an working tool in macrosituations they have proved their wide applicability, so it does not seem reasonable to leave them off fully and to search for new equations. Such a step would reject also the

corresponding well proved energetic relations for finite volumes, obtained by these equations. Therefore an appropriate their extension seems natural and reasonable provided that the new incorporated solutions have the desired properties.

In the non-vacuum case, i.e. in the presence of energy-momentum exchange between the field and some medium, there are also problems coming mainly from the *hypothesis* that the Faraday induction law is valid for *all* media, which leads to the notion that the field is able to exchange energy-momentum *only with electrically charged particles*. No doubt, such media exist, but the assumption that this is true for all media, we do not accept. Further, the usual way to treat the field through introducing the *polarization and magnetization* vectors and to describe the media by means of corresponding permeability tensors and bound currents and so on, may be good operational skills, but they do not lay on approved physical principles and do not generate new ideas. In this case, as well as in the vacuum case, we choose a different way, namely we extend the equations in such a way, that to keep everything, that classical electrodynamics is able to do, and to incorporate new solutions with good enough properties. Let's say quite clear, that our purpose in this case is the same as in the vacuum case, namely, *to describe soliton-like field configurations with sufficiently clear physical interpretation.*

In the both cases, the leading idea for extension of the equations is an analysis and appropriate formulation of the local energy-momentum conservation laws in relativistic terms. In result we obtain *nonlinear* equations. In the continuous medium case an additional and entirely new moment is considered, namely, a physical interpretation of the Frobenius integrability equations for Pfaff systems. Also, the classical concept of a continuous medium is extended and understood as an arbitrary continuous physical system, exchanging energy-momentum with the field. The encouraging results we obtain in the both cases give us reasons for hope that we are on the right way.

Finally, everyone, who decides to read this book, will follow the complete version of the author's mental way and will get to the results obtained following an easy and smooth way of reasoning. The real road, that I passed through, was very different from what is given here. There were many turns, unexpected traps and various positive and negative surprises, and all this was taking place, when the fashion in theoretical physics was quite different. The deep belief in the conservation laws, in their universal character, was the point of light, that was leading me through the jungle of unknown

possibilities and hard to evaluate hypotheses and helped me to withstand the falling on one after another fashionable topics. Writing this book I was doing my best to pay a maximum respect to every reader, to every positive opponent. And, in order not to appear any doubts about my great respect to the prominent men, who created Classical electrodynamics, I would like to give here the creed that was the basic stimulus during this period: *the respect and esteem paid to the creators can not be honest and genuine if they are not in correspondence with the respect and esteem paid to the Truth.*

Now, **Dear reader**, after you have got some idea of what could be learned from this book, and if you have already made up your mind to become well acquainted with the small harmonic world I tried to create, turn over this page and be my fair corrective up to the last line of the last page.

Chapter 1

From Classical Electrodynamics to Extended Electrodynamics

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1.1 Basic Notions and Equations of Classical Electrodynamics

1.1.1 Nonrelativistic approach

In the mathematical model of the electromagnetic phenomena, called shortly CLASSICAL ELECTRODYNAMICS (CED), the Newton's notions of space and time are assumed, namely, a mathematical model of the real space is the real 3-dimensional mathematical space $\mathcal{R}^3$, considered as a 3-dimensional Euclidean manifold with the standard Euclidean metric g, and a mathematical model of the real time is the 1-dimensional mathematical space $\mathcal{R}$. The standard orthogonal coordinates (x, y, z) are usually used, so that the metric tensor g_{ik} has canonical components, i.e. the diagonal elements are equal to 1, and all other components are zero. Then the isomorphism between the tangent vectors and the 1-forms, defined by g and denoted by the same letter g, (or g^{-1}), looks as follows:

$$g(V) = g\Big(V^i \frac{\partial}{\partial x^i}\Big) = V^i g_{ik} dx^k = V_k dx^k,$$

where $V_k = g_{ki}V^i$. This isomorphism is extended naturally (i.e. component-wise) for arbitrary tensors (or tensor fields). The metric g defines also a

volume element:

$$\omega_0 = \sqrt{|detg_{ik}|}dx \wedge dy \wedge dz = dx \wedge dy \wedge dz,$$

a covariant derivative ∇, which in standard coordinates reduces to the usual derivative, and the Hodge $*$-operator $* : \Lambda^p(\mathcal{R}^3) \rightarrow \Lambda^{3-p}(\mathcal{R}^3)$ according to the rule

$$\alpha \wedge \beta = g(*\alpha, \beta)\omega_0,$$

where α and β are p and $(3-p)$ forms respectively. The operator $*$ is linear and in the canonical orthonormal basis {dx,dy,dz} from the above defining relation we obtain

$$*1 = dx \wedge dy \wedge dz,\ *dx = dy \wedge dz,\ *dy = -dx \wedge dz,\ *dz = dx \wedge dy,$$

$$*dx \wedge dy = dz,\ *dx \wedge dz = -dy,\ *dy \wedge dz = dx,\ *dx \wedge dy \wedge dz = 1.$$

We note that the defining relation for the $*$ operator is equivalent to the following: if α and β are p-forms then

$$\alpha \wedge *\beta = g(\alpha, \beta)\omega_0.$$

If $\mathbf{d}$ is the *exteriour derivative*, then the operator $\delta = (-1)^p *^{-1} \mathbf{d}*$ is called *coderivative* and we get $(\delta\alpha)_{ik...} = -\nabla_j \alpha^j_{.ik...}$. It is easily checked, that the operator δ is the dual to $\mathbf{d}$ in the vector space of forms with compact support with respect to the inner product $(\alpha, \beta) = \int \alpha \wedge *\beta$, and for forms with different degree is assumed $(\alpha, \beta) = 0$.

From physical point of view CED assumes the following:

1. The Electromagnetic (EM) field is a real object (or a set of real objects) possessing energy and momentum.

2. The EM-field is able to interact (i.e. to exchange energy and momentum) with material particles, possessing the characteristic *electric charge* which takes positive, negative and zero values. Such charged particles are called *field sources*.

3. The sources are called

a)*free* - if under the action of the external field they can move throughout the whole space, and

b)*bounded* - if their motions are bounded inside comparatively small space regions.

In order to describe the EM-field CED introduces the following quantities:

I. A couple of vector fields (E, B), defined on $\mathcal{R}^3$, and a parametric dependence of (E, B) on the time t is admitted.

II. A scalar function $\rho(x, y, z, t)$, called *charge density*, such that the integral $\int \rho\omega_0$, computed over the region $V \subset \mathcal{R}^3$, gives the entire electric charge in V.

III. A vector field $\mathbf{v}(x, y, z, t)$, describing the mechanical motion of the charge-carriers, and the vector field $\mathbf{j} = \rho\mathbf{v}$ is called *electric current*.

Remark. Further a vector field V and the corresponding 1-form $g(V)$ will be denoted by the same latter, since from the context it will be clear if the object is tangent or co-tangent.

An important concept in CED is the so called *flow of a vector field U* through a given 2-dimensional surface S. By definition, it is the integral of the 2-form $*U$ over S: $\int_S *U$. It is important if the surface is closed (most frequently homeomorphic to the 2-sphere S^2), or it is not closed and has for a boundary a given 1-dimensional manifold.

As a generalization of the experimental data in CED is assumed the following:

1°. *The flows of the electric E and magnetic B fields through a closed 2-surface S^2, surrounding the 3-volume V are defined as follows*:

$$\oint_{S^2} *E = 4\pi \int_V \rho\omega_0, \quad \oint_{S^2} *B = 0.$$

2°. *If the 2-surface S is not closed and its boundary is the closed curve l, then the following relations hold*:

$$\frac{d}{dt}\int_S *E = c\int_l B - 4\pi\int_S *\mathbf{j}, \quad \frac{d}{dt}\int_S *B = -c\int_l E.$$

From these relations, making use of the notations $g^{-1} * \mathbf{d}g = rot$, $g^{-1}\delta g = div$ and the Stokes' formula

$$\int_S \mathbf{d}\alpha = \int_l \alpha$$

we get the differential equations

$$\frac{1}{c}\frac{\partial E}{\partial t} = rotB - \frac{4\pi}{c}\mathbf{j}, \quad divB = 0, \tag{1.1}$$

$$\frac{1}{c}\frac{\partial B}{\partial t} = -rotE, \qquad divE = 4\pi\rho. \tag{1.2}$$

Because of the identity $div_\circ rot = 0$, from the first equation of (1.1) and the second equation of (1.2) it follows the *continuity equation*

$$\frac{\partial \rho}{\partial t} = -div\mathbf{j}, \tag{1.3}$$

the sense of which becomes clear from its integral form

$$\frac{d}{dt}\int_V \rho\omega_0 = \oint_{S_V} *\mathbf{j}.$$

This relation shows, that any change of the electric charge inside the region V is caused by some processes of charge transport through the boundary of V, i.e. *charges do not vanish and do not arise.*

Using the scalar product g from the first equation of (1.1) and the first equation of (1.2) we obtain

$$\frac{1}{c}g\Big(E,\frac{\partial E}{\partial t}\Big)+\frac{1}{c}g\Big(B,\frac{\partial B}{\partial t}\Big) = g(E,rotB)-g(B,rotE)-\frac{4\pi}{c}g(E,\mathbf{j}).$$

Since g does not depend ot time and

$$g(E,rotB)-g(B,rotE) = -div(E\times B)$$

we get

$$\frac{\partial}{\partial t}\frac{E^2+B^2}{8\pi} = -\mathbf{j}.E - div\mathbf{S},$$

where the vector

$$\mathbf{S} = \frac{c}{4\pi}E\times B$$

is the Poynting vector. This last relation describes the local balance of the energy and momentum in the system *EM-field and free current* in an unit space-time volume.

The equations (1,1), (1,2) are linear, so that if (E_1,B_1) and (E_2,B_2) are two solutions, then any linear combination

$$E = aE_1 + bE_2,\ \ B = mB_1 + nB_2$$

with constant coefficients is again a solution. The following question arises naturally: do there exist constants (a,b,m,n), such that the linear combination

$$E' = aE + bB,\ \ B' = mE + nB$$

is again a solution? The answer to this question is positive iff $m = -b, n = a$. The new solution will have energy density w' and momentum $\mathbf{S}'$ as follows:

$$w' = \frac{1}{8\pi}\Big((E')^2 + (B')^2\Big) = \frac{1}{8\pi}(a^2+b^2)\Big(E^2 + B^2\Big),$$

$$\mathbf{S}' = (a^2+b^2)\frac{c}{4\pi}E \times B.$$

Obviously, the new and the old solutions will have the same energy and momentum if $a^2 + b^2 = 1$.

These observations show that besides the usual linearity, Maxwell's equations admit also "cross"-linearity, i.e. linear combinations of E and B of a definite kind define new solutions. Therefore, the difference between the electric and magnetic fields becomes non-essential. The important point is that with every solution (E, B) of Maxwell's equations a 2-dimensional real vector space, spanned by the couple (E, B), is associated, and the linear transformations, which transform solutions into solutions, are given by matrices of the kind

$$\left\| \begin{matrix} a & b \\ -b & a \end{matrix} \right\|.$$

If these matrices are unimodular, i.e. if $a^2 + b^2 = 1$, then the initial and the transformed solutions have the same energy and momentum. It is well known that matrices of this kind do not change the canonical complex structure J in $\mathcal{R}^2$: if the canonical basis of $\mathcal{R}^2$ is denoted by (e_1, e_2) then J is defined by $J(e_1) = e_2$, $J(e_2) = -e_1$.

The above remarks suggest to consider E and B as two vector-components of an $\mathcal{R}^2$-valued 1-form Ω:

$$\Omega = E \otimes e_1 + B \otimes e_2.$$

So, the current $\mathbf{j}$ becomes 1-form $\mathcal{J} = \mathbf{j} \otimes e_1$ with values in $\mathcal{R}^2$, and the charge density becomes a function $\mathcal{Q} = \rho \otimes e_1$ with values in $\mathcal{R}^2$. Maxwell's equations take the form

$$\frac{1}{c}\frac{\partial \Omega}{\partial t} = -\frac{4\pi}{c}\mathcal{J} - *\mathbf{d}J(\Omega), \quad \delta\Omega = 4\pi\mathcal{Q}, \tag{1.4}$$

where $J(\Omega) = E \otimes J(e_1) + B \otimes J(e_2) = E \otimes e_2 - B \otimes e_1$. Note that according to the sense of the concept of current in CED and because of the lack of magnetic charges, it is necessary to exist a basis of $\mathcal{R}^2$, in which $\mathcal{J}$ and $\mathcal{Q}$ to

have components only along e_1. Nevertheless, this point of view shows that even at this non-relativistic level the separation of the EM-field to "electric" and "magnetic" is not adequate to the real situation.

1.1.2 Relativistic formulation

We pass now to the relativistic formulation of CED. We begin with the note that the relativism, considered as a theoretical conception for understanding and modeling the natural entities and processes, arises as a result of the analysis of the invariance properties of Maxwell's equations with respect to linear transformations of space-time coordinates $(x, y, z, \xi = ct)$. The assumption for the linear character of the space-time transformations comes on the one side from the linearity of the model space $\mathcal{R}^4$ and Maxwell's equations, and on the other side, from the idea, that any straight-line uniform motion should not affect the character and the course of *all* physical processes. The latter formally means, that the parameters of the admissible space-time transformations, interpreted as straight-line uniform motions, should be determined by the relative constant velocity between two frames. So the cartesian framings in $\mathcal{R}^4$ model the physical inertial frames. The basic conclusion of the analysis carried out during the end of last and the beginning of the current century consists in, that the invariance of Maxwell's equations requires a *pseudo-eucledean space-time metric tensor η and, additionally, any component of E and B depends linearly on all components of E and B when a space-time transformation is performed.* (In connection with this we note, that the "cross"-linearity mentioned above admits "mixing" only for corresponding components of E and B: E_1 with B_1, E_2 with B_2 and E_3 with B_3.) This undoubtedly shows, that the adequate mathematical object, describing the EM-field, must have 6 independent components and its transformation properties should be determined entirely by the admissible space-time transformations. In view of the 4-dimensionality of space-time such objects are only the antisymmetric tensor fields of second order. Because of the possible identification of the contravariant and covariant tensor fields by means of the pseudo-metric tensor $\eta_{\mu\nu}$, the obvious candidates for models of the EM–field are the *differential 2–forms*. So, starting with the 3-dimensional vector fields E and B and with the pseudoeucledean structure $\eta_{\mu\nu}$ of the model space $\mathcal{R}^4$ we have to build a differential 2-form $F \in \Lambda^2(\mathcal{R}^4, \eta)$. Further, the space-time $(\mathcal{R}^4, \eta)$, where $\eta_{\mu\nu} = -1$ for $\mu = \nu = 1, 2, 3$; $\eta_{44} = 1$, and $\eta_{\mu\nu} = 0$ for $\mu \neq \nu$ in standard coordinates $(x^1, x^2, x^3, x^4) = (x, y, z, \xi = ct)$, will be denoted by

M and will be called Minkowski space. The pseudo-metric $\eta_{\mu\nu}$ defines in the same way a volume element ω_0 and the Hodge $*$-operator. We have

$$\omega_0 = \sqrt{|det\eta_{\mu\nu}|}dx \wedge dy \wedge dz \wedge d\xi,$$

$$\alpha \wedge \beta = \eta(*\alpha, \beta)\omega_0 \iff \alpha \wedge *\beta = -\eta(\alpha, \beta)\omega_0,$$

$$**_p = -(-1)^{p(4-p)}id, \; *_p^{-1} = -(-1)^{p(4-p)}*_p, \; *\omega_0 = 1, \; *1 = -\omega_0,$$

$$\begin{array}{ll}
*dx = dy \wedge dz \wedge d\xi & *dx \wedge dy \wedge dz = d\xi \\
*dy = -dx \wedge dz \wedge d\xi & *dx \wedge dy \wedge d\xi = dz \\
*dz = dx \wedge \; dy \wedge d\xi & *dx \wedge dz \wedge d\xi = -dy \\
*d\xi = dx \wedge dy \wedge dz & *dy \wedge dz \wedge d\xi = dx
\end{array}$$

$$\begin{array}{ll}
*dx \wedge dy = -dz \wedge d\xi & *dy \wedge dz = -dx \wedge d\xi \\
*dx \wedge dz = dy \wedge d\xi & *dy \wedge d\xi = -dx \wedge dz \\
*dx \wedge d\xi = dy \wedge dz & *dz \wedge d\xi = dx \wedge dy.
\end{array}$$

An arbitrary 2-form F on M in standard coordinates looks as follows:

$$F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu = F_{12}dx \wedge dy + F_{13}dx \wedge dz + F_{23}dy \wedge dz + \\ + F_{14}dx \wedge d\xi + F_{24}dy \wedge d\xi + F_{34}dz \wedge d\xi.$$

Then for $*F$ we obtain

$$* F = -\frac{1}{2}\varepsilon_{\mu\nu\sigma\tau}F^{\sigma\tau}dx^\mu \wedge dx^\nu = F_{34}dx \wedge dy - F_{24}dx \wedge dz + F_{14}dy \wedge dz - \\ - F_{23}dx \wedge d\xi + F_{13}dy \wedge d\xi - F_{12}dz \wedge d\xi.$$

The definition of the components $F_{\mu\nu}$ by means of the components of E and B is made in the following way. Let $i{:}\mathcal{R}^3 \to \mathcal{R}^4$ be the standard immersion $(x, y, z) \to (x, y, z, 0)$. Then we define i^*F and $i^*(*F)$ by

$$i^*F = *B, \quad i^*(*F) = *E, \tag{1.5}$$

where on the right-hand side of these equalities the Eucledean $*$–operator is used. Relations (1,5) define F uniquely, and we get

$$F_{12} = B_3, \; F_{13} = -B_2, \; F_{23} = B_1, \; F_{14} = E_1, \; F_{24} = E_2, \; F_{34} = E_3.$$

Recalling that in the static case with zero current Maxwell's equations reduce to $\mathbf{d}E = 0, \; \mathbf{d} * E = 0, \; \mathbf{d}B = 0, \; \mathbf{d} * B = 0$, and the well known relation

$\mathbf{d}i^* = i^*\mathbf{d}$, we obtain for this static case $i^*\mathbf{d}F = 0,\ i^*\mathbf{d} * F = 0$. The map i^* cancels all terms where $d\xi$ stays. Removing this map i^*, we get the equations $\mathbf{d}F = 0,\ \mathbf{d} * F = 0$, so we keep all terms with $d\xi$ non-canceled, and having in view the above component interpretation of $F_{\mu\nu}$ we obtain exactly the left-hand sides of Maxwell's equations (1,2), (1,1) respectively. We introduce now the 4-current j^μ by $j^\mu = \rho u^\mu$, where u^μ is the 4-velocity of the charged particles. Maxwell's equations take the form

$$\mathbf{d}F = 0, \quad \mathbf{d} * F = 4\pi * j. \tag{1.6}$$

These equations (1.6) may be written in various forms:

$$\delta * F = 0, \quad \delta F = 4\pi j$$

$$\frac{\partial F_{\mu\nu}}{\partial x^\sigma} + \frac{\partial F_{\nu\sigma}}{\partial x^\mu} + \frac{\partial F_{\sigma\mu}}{\partial x^\nu} = 0, \quad \nabla_\sigma F^{\sigma\nu} = -4\pi j^\nu.$$

Of definite importance for the theory is the quantity

$$Q_\mu^\nu = \frac{1}{4\pi}\left[\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\delta_\mu^\nu - F_{\mu\sigma}F^{\nu\sigma}\right] = \frac{1}{8\pi}\left[-F_{\mu\sigma}F^{\nu\sigma} - (*F)_{\mu\sigma}(*F)^{\nu\sigma}\right] \tag{1.7}$$

since on the solutions of (1.6) the following relation holds:

$$\nabla_\nu Q_\mu^\nu = \frac{1}{4\pi}\left[F_{\mu\nu}(\delta F)^\nu + (*F)_{\mu\nu}(\delta * F)^\nu\right] = F_{\mu\nu}j^\nu. \tag{1.8}$$

This relation describes the local energy-momentum balance in the system *EM–field and free current*. The quantity $F_{\mu\nu}j^\nu$ is the Lorentz force and it determines the energy-momentum, which the charge carriers get from the field in an unit 4-volume. In regions with zero current $j^\mu = 0$ we have $\nabla_\nu Q_\mu^\nu = 0$, so we may create integral conserved quantities. Because of its importance for the theory we shall consider this point more in detail.

In classical field theory we build integral conserved quantities, i.e. time-independent quantities, by means of a symmetric second rank tensor $Q_{\mu\nu}$ with zero divergence $\nabla_\nu Q_\mu^\nu = 0$ by making use of isometries, i.e. symmetries of the metric tensor, in the following way. By definition a symmetry of a tensor field g is a map $f : M \to M$, which keeps this tensor field unchanged: $f^*g = g$. When an one-parameter group of symmetries f_t, defined by the generator X, (or the vector field X) is given, then X is called *local symmetry* of g and the following relation holds

$$L_X g = lim_{t\to 0}\frac{f_t^* g - g}{t} = 0,$$

which means that the local symmetries of g are those vector fields X along the integral lines of which g stays unchanged. The expression on the right is called *Lie-derivative* of g along X. The local symmetries of the metric tensor are also called *Killing* vector fields. The equation $L_X g = 0$, where g is given, looks as follows

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0,$$

where ∇ is the corresponding symmetric Riemannean connection. If now $Q_{\mu\nu}$ is a conservative tensor field, i.e. $\nabla_\nu Q_\mu^\nu = 0$, and X is a local isometry, we obtain

$$\nabla_\nu (Q_\mu^\nu X^\mu) = (\nabla_\nu Q_\mu^\nu) X^\mu + Q^{\mu\nu} \nabla_\nu X_\mu = Q^{\mu\nu} \nabla_\nu X_\mu .$$

Because of the symmetry of Q, in the sum $Q^{\mu\nu}\nabla_\mu X_\nu$ only the symmetric part of $\nabla_\mu X_\nu$ may contribute, but this symmetric part is zero since X is a local isometry. In this way with every local isometry X of the metric the 1-form $Q_{\mu\nu}X^\mu dx^\nu$ is associated, and this 1-form has zero divergence. The last means that the 3-form $*(Q_{\mu\nu}X^\mu dx^\nu)$ is closed, so according to the Stokes theorem, the integral of this 3-form over $\mathcal{R}^3$ will not depend on time. Of course, these considerations make sense only for finite valued such 3-integrals, i.e. for *finite* field functions.

In order to complete the energy-momentum balance picture we have to point out how the charged particles of the 4-current use the gained from the field energy-momentum to change their behaviour, i.e. we have to write down the equations of motion of these charge carriers.

Assuming that only interaction between the particles and the field takes place, the energy-momentum tensor of the particles is defined by

$$K_\mu^\nu = \mu_o c^2 u^\nu u_\mu ,$$

where μ_o denotes the invariant mass density of the particles. So, the full local energy-momentum conservation law requires $\nabla_\nu (Q_\mu^\nu + K_\mu^\nu) = 0$. Since particles do not vanish and do not arise, which formally means that the mass continuity equation $\nabla_\nu(\mu_o u^\nu) = 0$ holds, we obtain

$$\mu_o c^2 u^\nu \nabla_\nu u_\mu = -F_{\mu\nu} j^\nu . \tag{1.9}$$

This equation for the vector field u describes the mechanical evolution of the charge-carriers. We note that this is a compact form of a *nonlinear* system of partial differential equations for the components of u, while Maxwell's

equations are linear. This shows that the current $j_\mu = \rho u_\mu$ *cannot be defined independently*, i.e. it strongly depends on the field F. Therefore, the variational procedure for verifying Maxwell's equations with "given and not subject to variation" current does not seem to be quite correct. Otherwise, we lose the full energy-momentum conservation law, which is hardly preferable.

Remark. It is clear that the particles with mass distribution μ_o and charge distribution ρ, are considered as *sources* of the field F according to the usual interpretation of equations (1.6) amd (1.7), and, therefore, they *cannot be considered as test particles*, i.e. as not disturbing the field. If the field is not disturbed, i.e. if it does not exchange energy-momentum with the particles, then $\nabla_\nu Q_\mu^\nu = 0$ and the corresponding solution of the field equations, which is meant to define "Lorentz force", has to satisfy the current-free Maxwell equations $\mathbf{d}F = 0$, $\delta F = 0$, as in the static Coulomb case. In this case the dynamical equations for the particles appear as an additional assumption, and talking about *full* energy-momentum conservation law is hardly sensible from the point of view of the field, nevertheless, it may get some sense from the point of view of the particles.

We consider now the *conformal invariance* of pure field Maxwell equations. In accordance with (1.6) if $j = 0$ we get

$$\mathbf{d}F = 0, \quad \mathbf{d} * F = 0. \tag{1.10}$$

The only factor not permitting a full invariance of these equations is the $*$-operator, more exactly, its restriction on 2-forms. On a 2n-dimensional riemannian manifold the restriction of $*$ on n-forms is always conformally invariant because 2 conformal metrics g and $\tilde{g} = f^2 g, f(x) \neq 0, x \in M$, generate the same $*_n$ operator. In our case $n = 2$ and $g = \eta$, so

$$\tilde{*}F = \frac{1}{2}F_{\mu\nu}\tilde{*}(dx^\mu \wedge dx^\nu) = -\frac{1}{2}F_{\mu\nu}\tilde{\eta}^{\mu\sigma}\tilde{\eta}^{\nu\tau}\varepsilon_{\sigma\tau\alpha\beta}\sqrt{|det\tilde{\eta}_{\rho\kappa}|}dx^\alpha \wedge dx^\beta =$$
$$-\frac{1}{2}F_{\mu\nu}f^{-4}\eta^{\mu\sigma}\eta^{\nu\tau}\varepsilon_{\sigma\tau\alpha\beta}f^4\sqrt{|det\eta_{\rho\kappa}|}dx^\alpha \wedge dx^\beta = *F.$$

If we recall the expression (1.7) for the energy-momentum tensor, we'll see that the $*$-operator is applied there also on 2-forms only, which shows, that it is conformally invariant too. Moreover, the expressions (1.7), (1.8) clearly show, that the following relations hold:

$$Q_\mu^\nu(F) = Q_\mu^\nu(*F), \quad \nabla_\nu Q_\mu^\nu(F) = \nabla_\nu Q_\mu^\nu(*F). \tag{1.11}$$

These two equalities determine a full equivalence from energetic point of view between F and $*F$. This important fact will be substantially used later when we'll be writing down the new equations of Extended Electrodynamics (EED).

From pure algebraic point of view the EM-field, i.e. the 2-form F, has two invariants

$$I_1 = \frac{1}{2}F_{\mu\nu}F^{\mu\nu} = B^2 - E^2, \quad I_2 = \frac{1}{2}F_{\mu\nu}(*F)^{\mu\nu} = 2B.E. \tag{1.12}$$

These are the coefficients (up to a sign) of the two 4-forms

$$F \wedge *F = -\eta(F,F)\omega_\circ = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\omega_\circ,$$

$$F \wedge F = -F \wedge **F = \eta(F,*F)\omega_\circ = \frac{1}{2}F_{\mu\nu}(*F)^{\mu\nu}\omega_\circ.$$

The following relations hold

$$(4\pi)^2 Q_{\mu\nu}Q^{\mu\nu} = I_1^2 + I_2^2, \quad (4\pi)^2 Q_{\mu\sigma}Q^{\nu\sigma} = \frac{1}{4}[I_1^2 + I_2^2]\delta_\mu^\nu.$$

The equations for the eigen values of F and $*F$

$$det\|F_{\mu\nu} - \lambda\eta_{\mu\nu}\| = 0, \quad det\|(*F)_{\mu\nu} - \lambda^*\eta_{\mu\nu}\| = 0$$

look as follows

$$\lambda^4 + I_1\lambda^2 - \frac{1}{4}I_2^2 = 0, \quad (\lambda^*)^4 - I_1(\lambda^*)^2 - \frac{1}{4}I_2^2 = 0.$$

For the corresponding eigen values we obtain

$$\lambda_{1,2} = \pm\sqrt{-\frac{1}{2}I_1 + \frac{1}{2}\sqrt{I_1^2 + I_2^2}}, \quad \lambda_{3,4} = \pm\sqrt{-\frac{1}{2}I_1 - \frac{1}{2}\sqrt{I_1^2 + I_2^2}},$$

$$\lambda^*_{1,2} = \pm\sqrt{\frac{1}{2}I_1 + \frac{1}{2}\sqrt{I_1^2 + I_2^2}}, \quad \lambda^*_{3,4} = \pm\sqrt{\frac{1}{2}I_1 - \frac{1}{2}\sqrt{I_1^2 + I_2^2}}.$$

Multiplying the equation $F^\mu_{.\nu}\xi^\nu = \lambda\xi^\mu$ on the left by $-F_\mu^{.\sigma}$ and adding on the two sides of the equation obtained $\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\delta_\mu^\nu$, we get

$$Q^\mu_\nu\xi^\nu = \gamma\xi^\mu, \quad \gamma = \left[\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \lambda^2\right] = \left[\frac{1}{2}I_1 + \lambda^2\right].$$

This shows, that the eigen vectors of F are eigen vectors of Q too, and the eigen values λ of F and γ of Q are related by the above condition. The corresponding relation between γ and λ^* reads

$$\gamma = \left[\frac{1}{4}(*F)_{\alpha\beta}(*F)^{\alpha\beta} + (\lambda^*)^2\right] = \left[-\frac{1}{2}I_1 + (\lambda^*)^2\right].$$

As for the eigen vectors of F, $*F$ and Q we mention just the *isotropic* case, i.e. when $I_1 = I_2 = 0$. Clearly, if this is the case, then all eigen values are equal to zero and it can be shown, that there exists *just one and common for F, *F, and Q isotropic eigen direction, defined by the isotropic vector* $\zeta, \zeta^2 = 0$, and all other eigen vectors are space-like. Moreover, there exist two 1-forms A and A^*, such that the following presentation holds

$$F = A \wedge \zeta, \quad *F = A^* \wedge \zeta. \tag{1.13}$$

Obviously, the 1-forms A and A^* are defined up to additive factors colinear to ζ. We show now that these two 1-forms are spacelike, mutually orthogonal, they have equal magnitudes and are orthogonal to ζ: $A^2 = (A^*)^2 < 0,\ A.A^* = 0,\ A.\zeta = A^*.\zeta = 0$. In fact,

$$0 = *(A^* \wedge A^* \wedge \zeta) = *(A^* \wedge *F) = -(A^*)^\sigma F_{\sigma\mu}dx^\mu = -(A^*)^\sigma A_\sigma \xi_\mu dx^\mu,$$

$$0 < 4\pi Q^4_4 = -F_{4\nu}F^{4\nu} = -(*F)_{4\nu}(*F)^{4\nu} = -A^2\zeta_4\zeta^4 = -(A^*)^2\zeta_4\zeta^4,$$

$$0 = I_1 = \frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(A_\mu\zeta_\nu - A_\nu\zeta_\mu)(A^\mu\zeta^\nu - A^\nu\zeta^\mu) =$$

$$-(A.\zeta)^2 = -\frac{1}{2}(*F)_{\mu\nu}(*F)^{\mu\nu} = (A^*.\zeta)^2.$$

We see that in this case of zero inzvariants $I_1 = I_2 = 0$ there exists a new invariant, namely the square of A and A^*. Besides, from the first row of equalities is seen that A is an eigen vector of $*F$ and A^* is an eigen vector of F. Here are two useful relations:

$$det\|F_{\mu\nu}\| = det\|(*F)_{\mu\nu}\| = \frac{1}{4}(I_2)^2, \quad det\|(F \pm *F)_{\mu\nu}\| = (I_1)^2. \tag{1.14}$$

Finally we note that all subdeterminants of third order are proportional to $E.B$, so, the highest order non-zero subdeterminants in this case may be those of second order.

1.1.3 Continuous media

All considerations made up to here are characterized by the assumption, that in regions with $\rho = 0$ the EM- field propagates in space without losing energy and momentum, i.e. there is no interaction with the medium, which we call in such a case *electromagnetic vacuum*. CED extends its applicability to media, which interact with the field, exchanging energy-momentum at every point and moment. The majority of the known really existing media are built of a great number of closely connected, i.e. strongly interacting, neutral and electrically charged particles. An exact description of the mechanical motion of each one of these particles in presence of the external EM-field is practically impossible, so we have to assume simplified models of the various media. Doing this, it is important not to forget all conditions and scales under which the simplifying assumptions of a given model work.

In order to adapt the already developed mathematical machinery of the theory to describe what happens in presence of such media, which are briefly called *macroscopic bodies*, the approximation *physically small volume* is introduced in the following way. If l denotes the average distance among the particles, creating a given medium, if ΔV denotes the physically small volume and if L denotes some typical linear scale of the macroscopic object, we want the following relations to hold

$$l^3 \ll \Delta V \ll L^3. \tag{1.15}$$

Further we assume these conditions satisfied, and all media, satisfying them will be shortly called *macromedia*.

From practical point of view important are those macromedia, which can be *electrified* and *magnetized* when placed in external EM-fields. Such media are called *dielectrics*. The additional electrifying is due to the presence of *bound charges* in these media. Subject to the action of the external field these charges perform limited in small regions displacements. Such displacements cause appearance of additional charges, currents and dipole moments. After an averaging over the volume ΔV, they are denoted respectively by ρ_b-*bound charge density*, $\mathbf{j}_b$-*bound current density*, and P-*polarization vector*. The additional magnetization is due to the circle-like displacements of the charges, generating in this way new magnetic moments. The averaging of these new magnetic moments over the volume ΔV defines the *magnetization vector* $\mathcal{M}$.

In analogy with the case *free charges in vacuum* the following relations

among these new quantities are assumed:

$$\rho_b = -divP, \quad \mathbf{j_b} = crot\mathcal{M} + \frac{\partial P}{\partial t}. \tag{1.16}$$

After replacing in Maxwell equations (1.1) and (1.2) $\mathbf{j}$ and ρ by $(\mathbf{j} + \mathbf{j}_b)$ and $(\rho + \rho_b)$ respectively, and having in view (1.16) we obtain the Maxwell's equations for continuous media

$$\frac{1}{c}\frac{\partial D}{\partial t} = rotH - \frac{4\pi}{c}\mathbf{j}, \quad divB = 0, \tag{1.17}$$

$$\frac{1}{c}\frac{\partial B}{\partial t} = -rotE, \qquad divD = 4\pi\rho, \tag{1.18}$$

where

$$H = B - 4\pi\mathcal{M}, \quad D = E + 4\pi P. \tag{1.19}$$

When passing from one medium to another, the dielectric properties of which strongly differ from those of the first one, it is naturally to expect a violation of the continuous properties of H and D. Therefore it is necessary to define the behaviour of these quantities on the corresponding boundary surfaces. To this end, two new quantities are introduced: *surface density of the electric charge* σ and *surface density of the current* i. Then the analysis of the above equations brings us to the following relations:

$$(D_n)_2 - (D_n)_1 = 4\pi\sigma, \quad (E_n)_2 - (E_n)_1 = 0,$$

$$(H_n)_2 - (H_n)_1 = \frac{4\pi}{c}i, \quad (B_n)_2 - (B_n)_1 = 0,$$

where the index "n" denotes the normal to the boundary surface component of the corresponding vector at some point.

Assuming that the quantity of electromagnetic energy, transformed to mechanical work or heat during 1sec. in the volume V is equal to $\int_V(\mathbf{j}.E)dV$, and making use of Maxwell's equations for medium, we get

$$(\mathbf{j}.E) = -\frac{1}{4\pi}\Big[\Big(E.\frac{\partial D}{\partial t}\Big) + \Big(H.\frac{\partial B}{\partial t}\Big)\Big] - div\Big[\frac{c}{4\pi}E \times H\Big]. \tag{1.20}$$

Replacing now $D = E + 4\pi P$ and $H = B - 4\pi M$ in this relation we obtain

$$(\mathbf{j}.E) = -\frac{\partial}{\partial t}\frac{E^2 + B^2}{8\pi} - div\Big[\frac{c}{4\pi}E \times B\Big] -$$

$$-\Big[E.\frac{\partial P}{\partial t}-\mathcal{M}.\frac{\partial B}{\partial t}\Big]+c.div\Big[E\times\mathcal{M}\Big].$$

These relations describe the local energy-momentum balance.

The 8 equations (1.17)-(1.18) have to determine 15 functions E_i, B_i, H_i, D_i, j_i. Clearly, more relations among these functions are needed, in order to determine them. The usual additional relations assumed are of the kind

$$P^i=P^i\Big(E^j,\frac{\partial E^j}{\partial x^k},...;B^j,\frac{\partial B^j}{\partial x^k},...\Big),\quad \mathcal{M}^i=\mathcal{M}^i\Big(E^j,\frac{\partial E^j}{\partial x^k},...;B^j,\frac{\partial B^j}{\partial x^k},...\Big).$$

The most frequently met assumption is $P=P(E),\ \mathcal{M}=\mathcal{M}(B)$ together with the requirement $P(0)=0,\ \mathcal{M}(0)=0$. A series development gives

$$P^i=\kappa^i_jE^j+\frac{1}{2}\kappa^i_{jk}E^jE^k+...;\quad \mathcal{M}^i=\alpha^i_jB^j+\frac{1}{2}\alpha^i_{jk}B^jB^k+...$$

The tensors $\kappa^i_j,\ \kappa^i_{jk}, ...$ are called *polarization tensors* (of corresponding rank), and $\alpha^i_j, \alpha^i_{jk}, ...$ are called *magnetization tensors* (of corresponding rank). For D^i and H^i we obtain respectively

$$D^i=E^i+4\pi(\kappa^i_jE^j+\frac{1}{2}\ \kappa^i_{jk}E^jE^k+...)=(\delta^i_j+4\pi\kappa^i_j)E^j+...$$

$$H^i=B^i-4\pi(\alpha^i_jB^j+\frac{1}{2}\alpha^i_{jk}B^jB^k+...)=(\delta^i_j-4\pi\alpha^i_j)B^j-...$$

If the medium is homogeneous and isotropic and the EM-field is weak, the nonlinearities in these developments are neglected, so

$$D^i=(1+4\pi\kappa)\delta^i_jE^j=\varepsilon^i_jE^j=\varepsilon\delta^i_jE^j$$

and

$$H^i=(1-4\pi\alpha)\delta^i_jB^j=\alpha^i_jB^J=\alpha\delta^i_jB^j.$$

The constants ε and $\mu=\alpha^{-1}$ are called *dielectric* and *magnetic* permeabilities respectively. In case of nonisotropic media the two tensors ε^i_j and $\mu^i_j=(\alpha^i_j)^{-1}$ are used.

In the relativistic formulation of the electrodynamics of continuous media besides the 2-form F, a new 2-form S is introduced, namely

$$S=\mathcal{M}_3dx\wedge dy-\mathcal{M}_2dx\wedge dz+\mathcal{M}_1dy\wedge dz-P_1dx\wedge d\xi-P_2dy\wedge d\xi-P_3dz\wedge d\xi$$

as well as a new current

$$j_b^\mu = (\frac{1}{c}\mathbf{j}_b, \rho_b).$$

With these notations the equations (1.16) acquire the following compact form

$$\frac{\partial S^{\sigma\nu}}{\partial x^\sigma} = -J_b^\nu. \tag{1.21}$$

If we introduce now the 2-form $G = F - 4\pi S$, then equations (1,17)-(1,18) look as follows

$$\delta * F = 0, \quad \delta G = 4\pi j. \tag{1.22}$$

The two relations $D^i = \varepsilon^i_j E^j$ and $B^i = \mu^i_j H^j$ may be unified in one relation of the kind

$$G_{\mu\nu} = R^{..\alpha\beta}_{\mu\nu} F_{\alpha\beta},\ \mu < \nu,\ \alpha < \beta. \tag{1.23}$$

Obviously, $R^{..\alpha\beta}_{\mu\nu} = -R^{..\alpha\beta}_{\nu\mu}$, $R^{..\alpha\beta}_{\mu\nu} = -R^{..\beta\alpha}_{\mu\nu}$. Comparing now the relativistic relation (1.23) and the non-relativistic two relations, we obtain

$$R^{..kl}_{i4} = 0,\ R^{..j4}_{kl} = 0,\ R^{..j4}_{i4} = \varepsilon^j_i,$$

$$R^{..mn}_{kl} = \tilde{\varepsilon}_{klr}\chi^r_s\tilde{\varepsilon}^{smn},\ \chi^r_s = (\mu^r_s)^{-1},\ k < l,\ m < n.$$

The equations $\varepsilon^i_j = \varepsilon^j_i$, $\mu^i_j = \mu^j_i$ lead to $R^{..\alpha\beta}_{\mu\nu} = R^{..\mu\nu}_{\alpha\beta}$. It is immediately verified that

$$R^{..\alpha\beta}_{\mu\nu} + R^{..\beta\nu}_{\mu\alpha} + R^{..\nu\alpha}_{\mu\beta} = 0.$$

The (6×6) matrix $R^{..\alpha\beta}_{\mu\nu}$ looks as follows:

$$R^{..\alpha\beta}_{\mu\nu} = \left\| \begin{array}{cccccc} \chi^3_3 & -\chi^3_2 & \chi^3_1 & 0 & 0 & 0 \\ -\chi^2_3 & \chi^2_2 & -\chi^2_1 & 0 & 0 & 0 \\ \chi^1_3 & \chi^1_2 & \chi^1_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon^1_1 & \varepsilon^2_1 & \varepsilon^3_1 \\ 0 & 0 & 0 & \varepsilon^1_2 & \varepsilon^2_2 & \varepsilon^3_2 \\ 0 & 0 & 0 & \varepsilon^1_3 & \varepsilon^2_3 & \varepsilon^3_3 \end{array} \right\|.$$

For the invariant $R = R^{..\mu\nu}_{\mu\nu}$ we obtain

$$R = 2(\varepsilon^1_1 + \varepsilon^2_2 + \varepsilon^3_3 + \chi^1_1 + \chi^2_2 + \chi^3_3).$$

These algebraic properties of the tensor $R^{..\alpha\beta}_{\mu\nu}$ are the same as those of the Riemann curvature tensor. Since for vacuum we have $\varepsilon^j_i = \chi^j_i = \delta^j_i$ for $R^{..\alpha\beta}_{\mu\nu}$ we get

$$R^{..\alpha\beta}_{\mu\nu} = \delta^\alpha_\mu\delta^\beta_\nu - \delta^\beta_\mu\delta^\alpha_\nu,$$

or

$$R_{\mu\nu,\alpha\beta} = \eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\beta}\eta_{\nu\alpha},$$

which is exactly the induced by η metric in the bundle of 2-forms over the Minkowski space-time. In other words, the presence of an active continuous medium, i.e. non-trivial functions $\varepsilon_i^j(x^\nu)$ and $\chi_i^j(x^\nu)$, could be described by an appropriate *curved* metric in the space $\Lambda^2(M)$.

Now we are going to consider the problem of energy-momentum distribution of the field in presence of an active medium. Recall that in case of vacuum, these quantities are described by the energy-momentum tensor

$$Q_\mu^\nu = \frac{1}{4\pi}\left[\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\delta_\mu^\nu - F_{\mu\sigma}F^{\nu\sigma}\right] = \frac{1}{8\pi}\left[-F_{\mu\sigma}F^{\nu\sigma} - (*F)_{\mu\sigma}(*F)^{\nu\sigma}\right].$$

The natural generalization of this tensor in presence of a new 2-form S, or G, looks as follows

$$W_\mu^\nu = \frac{1}{8\pi}\left[\frac{1}{2}F_{\alpha\beta}G^{\alpha\beta}\delta_\mu^\nu - F_{\mu\sigma}G^{\nu\sigma} - G_{\mu\sigma}F^{\nu\sigma}\right]. \tag{1.24}$$

Using the identity, which holds for any two 2-forms in the Minkowski space

$$\frac{1}{2}F_{\alpha\beta}G^{\alpha\beta}\delta_\mu^\nu = F_{\mu\sigma}G^{\nu\sigma} - (*G)_{\mu\sigma}(*F)^{\nu\sigma}, \tag{1.25}$$

for W_μ^ν is obtained

$$W_\mu^\nu = \frac{1}{8\pi}\left[-F_{\mu\sigma}G^{\nu\sigma} - (*F)_{\mu\sigma}(*G)^{\nu\sigma}\right] = \frac{1}{8\pi}\left[-G_{\mu\sigma}F^{\nu\sigma} - (*G)_{\mu\sigma}(*F)^{\nu\sigma}\right].$$

Obviously, $W_{\mu\nu} = W_{\nu\mu}$, and if $S_{\mu\nu} \to 0$, or equivalently, $G = F$, we get $W_\mu^\nu \to Q_\mu^\nu$. Here are the explicit expressions of W_μ^ν by means of the components of the 3-vectors E, B, D, H:

$$W_i^j = \frac{1}{8\pi}\left[E_iD_j + E_jD_i + B_iH_j + B_jH_i + \delta_i^j(B.H - E.D)\right],$$

$$W_i^4 = \frac{1}{8\pi}\left[(E\times H)_i + (B\times D)_i\right], \quad W_4^4 = \frac{1}{8\pi}(E.D + B.H)$$

It is easily verified the following relation

$$\nabla_\nu W_\mu^\nu = \frac{1}{8\pi}\left[F_{\mu\nu}(\delta G)^\nu + G_{\mu\nu}(\delta F)^\nu + (*F)_{\mu\nu}(\delta{*G})^\nu + (*G)_{\mu\nu}(\delta{*F})^\nu\right]. \tag{1.26}$$

If we require at $j = 0$ in (1.22) the following local conservation law to hold

$$\nabla_\nu W_\mu^\nu = 0 \tag{1.27}$$

and make use of the above introduced definitions, we get the equation

$$4\pi S_{\mu\nu}(\delta S)^\nu = F_{\mu\nu}(\delta S)^\nu - (*F)_{\mu\nu}(\delta * S)^\nu. \tag{1.28}$$

This relation determines in what way and how much of the field energy-momentum is utilized by the medium, characterized by $S_{\mu\nu}$. Since the equations are 8, and the unknown functions are 12, the 4 equations (1.28), although *nonlinear*, could be used as additional field equations for $F_{\mu\nu}$ and $S_{\mu\nu}$. In this way the number of the equations becomes equal to the number of the unknown functions.

As we mentioned earlier, the symmetry of the energy-momentum tensor W_μ^ν is necessary to define correctly integral conserved quantities, using the 10 Killing vector fields of Minkowski space. The tensor W_μ^ν, given by (1.24), has the following additional properties:

$$W(F,G) = W(G,F), \quad W(F,G) = W(*F,*G),$$

$$\nabla_\nu W_\mu^\nu(F,G) = \nabla_\nu W_\mu^\nu(G,F), \quad \nabla_\nu W_\mu^\nu(F,G) = \nabla_\nu W_\mu^\nu(*F,*G).$$

Moreover, since the $*$-operator is applied only on 2-forms, the tensor W_μ^ν is *conformally invariant*. Finally, if the free current j is not equal to zero, then from (1.26) and (1.28) and Maxwell's equations it follows that the energy-momentum, transferred over to the medium in an unit 4-volume is determined by the well known expression $F_{\mu\nu}j^\nu$. We should not forget also, that if $G_{\mu\nu} = F_{\mu\nu}$ we obtain $W_\mu^\nu = Q_\mu^\nu$.

All these properties of W_μ^ν suggest that it is a good candidate for energy-momentum tensor of the EM-field in presence of a continuous medium. As for the new condition (1.27), its adequacy to the real processes of energy-momentum redistribution in the medium under consideration has to be verified every time we are going to use it.

1.2 Solutions to Maxwell's Equations

1.2.1 The solutions as models of physical objects and processes

The equations are mathematical relations among several quantities, some of which are known and some are unknown. To solve an equation means to find those values of the unknown quantities, which make this equation into identity. Clearly, the values found, i.e. the solutions, will depend on the values of the known quantities, as well as on the very equation. If the number of the independent equations is not equal to the number of the unknowns we say that the problem is *underdetermined* or *overdetermined* depending on whether the number of equations is *less* or *larger* than the number of the unknowns respectively. One of the important problems is that of *uniqueness*, i.e. under what conditions a given equation has only one solution. This problem is based on the fact that an equation may have many, even infinitely many, solutions. For example, the equation $f'(x) = 0$ is satisfied by every constant function; the equation $\frac{\partial f}{\partial x} = 0$, where the unknown function f depends on two independent variables (x, y), has infinitely many solutions, which are parametrized by one differentiable function of one variable: $f(x, y) = g(y)$.

In mathematics we frequently write down relations, which have sense in various classes of functions, even in various classes of mathematical objects. For example, the equation $\mathbf{d}\alpha = 0$ admits as solutions differential forms of various degree as well as differential forms with compact or noncompact support on various manifolds. This is true for a very large class of differential operators, therefore, it is necessary the set of objects, where we are going to look for a solution, to be pointed out sufficiently in detail. It may happen, that in the set of objects, where we look for a solution, the equation considered has no solutions. For example, the widely used Laplace equation $\Delta f = 0$ has no non-constant solutions in the class of *finite* functions $f : \mathcal{R}^3 \rightarrow \mathcal{R}$; the widely used *wave* equation $\Box f = 0$ has infinitely many (1+1)-soliton-like solutions, and *has no* (3+1)-soliton-like solutions.

So, every differential equation separates a class of functions (or objects), namely those, which have the *local* property to satisfy this differential equation. Not every solution has properties desired by us, so any separation of some subclass of solutions with definite additional properties is made by means of imposing additional conditions.

In theoretical physics we make *mathematical models of physical objects and processes.* We assume the idea, that *the really existing objects are finite and time-stable.* This means, that at any moment of their life they occupy

finite and sufficiently small regions of the 3-dimensional space and if there are no perturbations from outside, they would live sufficiently long time, i.e. they are time-stable. From mathematical point of view this requires that, with respect to the space variables (x, y, z) the corresponding mathematical objects *are everywhere smooth and have compact support*. The time evolution of the object is usually determined by an equation, defining a definite relation among the various derivatives of the components of the corresponding mathematical object with respect to the time and space coordinates, i.e. *locally*.

The real physical objects have *structural properties*, as well as properties as a whole, i.e. *integral* properties. If we are interested only in the integral properties and behaviour of the physical object, we talk about *material points*, and the behaviour of such objects is described by *ordinary differential equations*. The local description of an object, having a dynamical structure, requires *partial differential equations*, as well as rules, pointing out the connection between the local and integral characteristics and properties of the object.

If we are able to identify a physical object during a finite period of time, this means, that with respect to our means of identification this object shows *definite properties of constancy*. If these properties of constancy are measurable, then the introduced by corresponding measurement procedures quantities will have constant in time values, so we can talk about *conservation laws*, which are specific for the object considered. There exist physical quantities, which can be introduced for various physical objects. The importance of a physical quantity is in a direct dependence on its *universality*, i.e. how much broad is the class of objects, admitting this quantity as a characteristic.

At a definite level and under definite conditions it is possible some objects to be transformed into other objects. If such a transformation takes place the natural question arises:*what is the behaviour of those physical quantities, which characterize the initial objects, as well as the final ones, is there any quantitative connection between the initial and final values of these quantities?*. A positive response to this question would be of great importance for theoretical physics since it would allow some definite mathematical relations to be written down immediately and some *true* conclusions, based on these relations and concerning the real objects and processes, to be made.

The study of the real objects and processes from this point of view has resulted into formulation of the so called *conservation principles for energy, momentum and the full angular momentum.* (A principle is any rule which

is proved to hold for all known situations and which is extended as a hypothesis for the unknown situations). The physicists have succeeded in defining these quantities in such a way, that the time constancy of their values to be an adequate expression of the constancy properties of the real objects and to point out definite necessary conditions, which all real processes and transformations of objects have to respect and obey. We would like to note specially, that the *universality* of these quantities and principles is of primary importance.

Another very important property of these quantities is their *additiveness*, which enables us to build integral conservation laws, making use of the corresponding local conservation laws, i.e. time independent characteristics of the system as a whole.

Let us now connect these conclusions with the above mentioned circumstance, that the partial differential equations, which we use to describe the existence and evolution of the extended natural objects, admit usually many solutions. Most frequently, the model differential equation is a mathematical expression of one or several characteristic or typical local features of the object (or process), but not of all of them. Therefore, some of the solutions may posses properties, which do not correspond to the real integral properties of the object under consideration. So, additional conditions, such as new equations, inequalities, etc., of local or integral character have to be formulated. In such a case, in order to separate the non-adequate solutions, we shall always try to combine the above introduced notion for *finite and time stable object* and the availability of conserved quantities for any object, observing the following rule:

In order some solution of a given differential equation to be considered as a realistic model of a real object it is necessary this solution to be finite, time-stable and the corresponding integral energy, momentum and full angular momentum to be well defined finite quantities.

1.2.2 Spherically symmetric solutions

As we mentioned in section 1.1.2 the symmetry of a tangent object (i.e. a section of some tensor degree of the tangent and cotangent bundles) is a smooth map (usually a diffeomorphism) of the base space onto itself, such that the object does not change. We recall that in the non-relativistic frame

the base space is $\mathcal{R}^3$, while in the relativistic frame the base space is $\mathcal{R}^4$. Since the spherically symmetric solutions of Maxwell's equations are very important and lead to the notion of *elementary charge*, we are going to consider this point more in detail in nonrelativistic, as well as, in relativistic terms.

The intuitive notion of spherical symmetry consists in, that at the same distance from a given point the properties of the object under consideration are the same. The more accurate notion in the frame of the mathematical model requires:

1. Pointing out the point of the base manifold, which is the center of symmetry.

2. A definition of *distance*.

A procedure for separation of those maps, which keep the symmetry center stable as well as the distance between any two points unchanged.

In the frame of non-relativistic considerations the base space $\mathcal{R}^3$, is furnished by the Eucledean distance Δl between any two points with standard coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2)

$$\Delta l = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2},$$

or, equivalently, by the metric tensor g, which in these coordinates has the well known canonical components $g_{ii} = 1, g_{ij} = 0 \ if \ i \neq j$. Then the local symmetries$X$ of g are determined by the relation $L_X g = 0$. This last relation defines a system of differential equations for the components of the vector field X. Besides the standard translations $[\partial/\partial x, \partial/\partial y, \partial/\partial z]$, these equations have in spherical coordinates (r, θ, φ) the following solutions:

$$X_1 = \sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi},\ X_2 = -\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi},\ X_3 = -\frac{\partial}{\partial\varphi}.$$

These, of course, are the generators of the group $SO(3)$. So, we want our EM-field, i.e. the couple (E, B) of vector fields, to be symmetric with respect to the local isometries X_1, X_2, X_3, i.e. the following equations to hold

$$L_{X_i} E = [X_i, E] = 0,\ L_{X_i} B = [X_i, B] = 0$$

The solutions of these equations are

$$E = q(r;t)\frac{\partial}{\partial r},\ B = m(r;t)\frac{\partial}{\partial r}.$$

Since $g_{11} = 1$, then the corresponding 1-forms are $E = q(r;t)dr$, $B = m(r;t)dr$. Obviously, $\mathbf{d}E = 0$, $\mathbf{d}B = 0$, i.e. $rotE = 0$, $rotB = 0$. From Maxwell's equations with zero current it follows now that E and B do not depend on time, i.e. *all spherically solutions of the current-free Maxwell's equations are static.*

Let's now pay some attention to the following intermediate result: Every spherically symmetric, i.e. $SO(3)$- invariant , 1-form α on $\mathcal{R}^3$ is of the kind $\alpha = f(r)dr$, so it is closed: $\mathbf{d}\alpha = 0$. Then, if ω is $SO(3)$-invariant 2-form on $\mathcal{R}^3$, and since the action of the isometries commutes with the $*$-operator, defined by the same metric, we conclude that $*\omega$ is also $SO(3)$-invariant, hence it is closed. Historically, the things have started with a specially chosen 1-form, namely the Coulomb force F for a unit test charge, so it necessarily it is closed, and the special properties of F come from the observation that $*F$ is also closed. From modern point of view the $SO(3)$-invariant 2-form approach seems more natural and clearer.

What has rest to be done is to solve the two equations $\mathbf{d}*E = 0$, $\mathbf{d}*B = 0$. For the $*$-operator in spherical coordinates we get

$$\mathbf{d}*E = \mathbf{d}*q(r)dr = \mathbf{d}[q(r)r^2 \sin\theta d\theta \wedge d\varphi] = \frac{\partial(qr^2)}{\partial r}\sin\theta dr \wedge d\theta \wedge d\varphi = 0.$$

We obtain $q(r) = const/r^2 = q_0/r^2$, so

$$E = \frac{q_0}{r^2}\frac{\partial}{\partial r},\ B = \frac{m_0}{r^2}\frac{\partial}{\partial r}.$$

These results may be given a topological interpretation. In fact, the differential forms

$$E = \frac{q_0}{r^2}dr, \quad *E = q_0 \sin\theta d\theta \wedge d\varphi$$

are defined on the space $\mathcal{R}^3 \setminus \{0\}$, i.e out of the point $r = 0$. Maxwell's equations require both of them to be closed, but while the 1-form $E = q_0/r^2 dr$ is exact, the two form $*E = q_0 sin\theta d\theta \wedge d\varphi$ is not exact, since $\int_{S^2} *E = q_0$. Therefore, the 2-form $*E$ represents the nontrivial cohomology class of the space $\mathcal{R}^3 \setminus \{0\}$, moreover, it is the only spherically symmetric representative of this class.

If now we favour the topological character of the field configuration obtained, we should define the electric charge as the (appropriately parametrized) cohomology class of the space $\mathcal{R}^3 \setminus \{0\}$, and the field, as a representative of this class (we chose the spherically symmetric representative

$\omega = q_0 sin\theta d\theta \wedge d\varphi$ of this class, but this is not obligatory). The available eucledean metric g makes it possible to introduce the $*$-operator and the notion of spherical symmetry. Then the 1-form $*\omega = q_0 r^{-2}dr$, which now depends on the metric chosen, is exact and coincides with the notion of a field of an elementary source in CED. Note that, if we choose another metric $\tilde{g}$ on $\mathcal{R}^3 \setminus \{0\}$, then the relation $\mathbf{d} * \omega = 0$ may not hold. So, the notion of field in CED is strongly connected with the eucledean metric, while the notion of charge has topological origin.

This picture favours the topological aspect of the electric charge and gives some benefits for the relativistic formulation of CED. In fact, the closed 2-form ω is pulled back naturally as a closed 2-form on $(\mathcal{R}^3 \setminus \{0\}) \times \mathcal{R}$, and the pseudoeucledean metric η guarantees that $*_\eta \omega$ is closed, even exact. Of course, the consideration of all solutions of the two equations $\mathbf{d}\omega = 0$, $\mathbf{d} * \omega = 0$ as admissible models of real EM-fields is an additional hypothesis, but it is a natural one in the frame of this approach.

In the frame of the relativistic formalism the spherically symmetrical problem requires to solve the equations $\mathbf{d}F = 0,\ \mathbf{d} * F = 0$ together with the symmetry relations $L_{X_i}F = 0,\ L_{X_i} * F = 0$, which reduce to the first relation only since the $*-$operator is spherically symmetric. The most general form of a spherically symmetric F looks as follows:

$$F = f(r,\xi)dr \wedge d\xi + h(r,\xi)\sin\theta d\theta \wedge d\varphi.$$

Maxwell's equations require

$$F = \frac{C_1}{r^2}dr \wedge d\xi + C_2 \sin\theta d\theta \wedge d\varphi,$$

where C_1 and C_2 are constants. The additional requirements

$$F_{r\rightarrow\infty} \rightarrow 0, \qquad \oint_{S^2} *F = q_0$$

determine $C_1 = q_0/4\pi,\ C_2 = 0$.

1.2.3 Local and integral interaction energy of spherically symmetric fields. The Coulomb's force

Let us consider the following situation: two electric charges, occupying two 3-dimensional regions $\{O_1\}$ and $\{O_2\}$, considered as "open balls". The boundaries of these balls are the two non-overlapping spheres S_1 and S_2, and the

distance between the centers of these two spheres is R. The non-trivial topology of the space, where the two representatives ω_1 and ω_2 of the corresponding cohomology classes are defined is $\mathcal{R}^3 \setminus \{O_1, O_2\}$. Our purpose is to define *interaction* between the two fields ω_1 and ω_2 in such a way, that the calculated integral interaction energy to be equal to the well known classical expression $W = q_1 q_2/r$. Clearly, the local interaction energy expression has to be a 3-form and symmetric with respect to the two interacting fields. We define this 3-form as follows:

$$w = \frac{1}{4\pi}\omega_1 \wedge *\omega_2. \tag{1.29}$$

The two fields ω_1 and ω_2 are the only spherically symmetric representatives of the two cohomology classes. Since the eucledean metric is also spherically symmetric, the corresponding 1-forms $*\omega_1$ and $*\omega_2$ are also closed 1-forms and even exact. Therefore, there are two functions f_1 and f_2, such that $\mathbf{d}f_1 = *\omega_1$ and $\mathbf{d}f_2 = *\omega_2$. The 3-form w turns out to be exact, in fact

$$4\pi w = \omega_1 \wedge *\omega_2 = \frac{1}{2}\omega_1 \wedge *\omega_2 + \frac{1}{2}\omega_2 \wedge *\omega_1 =$$

$$= \frac{1}{2}[\omega_1 \wedge \mathbf{d}f_2 + \omega_2 \wedge \mathbf{d}f_1] = \frac{1}{2}[\mathbf{d}(f_2\omega_1) + \mathbf{d}(f_1\omega_2)] = \frac{1}{2}\mathbf{d}[f_2\omega_1 + f_1\omega_2].$$

We integrate now this expression over the region D out of the two open balls. Obviously, the boundary S_D of D is $S_D = S^2_\infty \cup S_1 \cup S_2$. Then, making use of the Stokes theorem, we obtain

$$W = \int_D w = \frac{1}{8\pi}\int_D \mathbf{d}[f_2\omega_1 + f_1\omega_2] =$$

$$= -\frac{1}{8\pi}\Big[\int_{S^2_\infty}(f_1\omega_2 + f_2\omega_1) \ + \ \int_{S_1}(f_1\omega_2 + f_2\omega_1) \ + \ \int_{S_2}(f_1\omega_2 + f_2\omega_1)\Big].$$

We introduce two spherical coordinate systems (r, θ, φ) and $(\bar{r}, \bar{\theta}, \bar{\varphi})$ originating at the centers of S_1 and S_2. We can write

$$\omega_1 = q_1 \sin\theta d\theta \wedge d\varphi, \quad f_1 = -\frac{q_1}{r},$$

$$\omega_2 = q_2 \sin\bar{\theta} d\bar{\theta} \wedge d\bar{\varphi}, \quad f_2 = -\frac{q_2}{\bar{r}}$$

Since f_1 and f_2 get zero values on the infinite sphere, we have to calculate the integrals over the two spheres S_1 and S_2. Having in view the relations

$$\int_{S_2}\omega_1=0,\ \int_{S_1}\omega_2=0,\ f_1|_{S_1}=const,\ f_2|_{S_2}=const$$

we obtain

$$W=-\frac{1}{8\pi}\Big[\int_{S_1}f_2\omega_1+\int_{S_2}f_1\omega_2\Big]=\frac{q_1q_2}{8\pi}\Big[\int_{S_1}\frac{\sin\theta d\theta\wedge d\varphi}{\bar r}+\int_{S_2}\frac{\sin\bar\theta d\bar\theta\wedge d\bar\varphi}{r}\Big]=$$

$$=\frac{q_1q_2}{8\pi}\Big[\frac{1}{R_1^2}\int_{S_1}\frac{dS_1}{\bar r}+\frac{1}{R_2^2}\int_{S_2}\frac{dS_2}{r}\Big],$$

where $dS_1=R_1^2\sin\theta d\theta\wedge d\varphi$ and $dS_2=R_2^2\sin\bar\theta d\bar\theta\wedge d\bar\varphi$ are the two surface elements. Since the function $1/r$ is *harmonic* out of the point $r=0$, then using the mean-value theorem for a harmonic function, we get

$$\frac{1}{4\pi R_1^2}\int_{S_1}\frac{dS_1}{\bar r}=\frac{1}{R},\quad \frac{1}{4\pi R_2^2}\int_{S_2}\frac{dS_2}{r}=\frac{1}{R}.$$

Finally,

$$W=\frac{q_1q_2}{R},$$

in correspondence with the purpose we set. This result shows that the differential $\mathbf{d}W$, where W is considered as a function of the two centers $r_1=0,r_2=0$, has nothing to do with $*\omega_1$, or $*\omega_2$, which are not defined at these points at all.

In order to introduce the Coulomb's force in a correct way we first note that its real sense is to define what part of the full energy and momentum of the two fields is transformed into mechanical energy and momentum of the two particles as a consequence of the local interaction of the two fields. This is an integral effect, since the very concept of electric charge has an integral character. Formally, we attack this problem in the following way. First we consider the trivial bundle $(\mathcal{R}^3\times q_1S^2,\pi,\mathcal{R}^3,S^2)$. In the natural coordinates $(x,y,z;\theta,\varphi)$ we consider the 3-form

$$\Omega_{12}=q_1sin\theta d\theta\wedge d\varphi\wedge\pi^*(*\omega_2)=q_1sin\theta d\theta\wedge d\varphi\wedge\frac{q_2(xdx+ydy+zdz)}{\sqrt{(x^2+y^2+z^2)^3}},$$

defined on the whole bundle space. Now the Coulomb's force can be defined as the *fiber integral* of Ω_{12} as follows:

$$\int_{S^2}\Omega_{12}=q_1q_2\frac{(xdx+ydy+zdz)}{\sqrt{(x^2+y^2+z^2)^3}}$$

The so obtained Coulomb's force could be interpreted as a characteristic of the field ω_2 when $q_1=1$, but it shouldn't be identified in any way with $*\omega_2$, although they look the same.

Let's consider briefly the problem of general spherically symmetric solutions of Maxwell's equations. Since in this case the time derivatives vanish we have the system

$$rotE=0,\ rotB=0,\ divE=0,\ divB=0,$$

or, all the same

$$\mathbf{d}E=0,\ \mathbf{d}B=0,\ \delta E=0,\ \delta B=0.$$

Now, the Laplace operator Δ is defined by

$$\Delta=\mathbf{d}\delta+\delta\mathbf{d},$$

so, we get

$$\Delta E=0,\ \Delta B=0.$$

(We should not forget that these are just *necessary* conditions, i.e. the last two equations may have solutions, which do not satisfy the static Maxwell equations.) We see, that the components of E and B are *harmonic* functions. According to the theory of these functions, they can not have local exteremums inside the regions of harmonicity, and if the region is the whole space $\mathcal{R}^3$ and the function is in addition bounded, it is a *constant*. These two properties of the solutions of Laplace equation do not recommend them as possible models of free real objects, which are finite and have to be described by finite functions of the three space variables (x,y,z), i.e. such, that necessarily to achieve their maximum values, since they are zero out of some finite subregions of $\mathcal{R}^3$. Only in some topologically non-trivial regions they could be of some interest in this respect.

Finally we note, that from relativistic point of view the concept "static field" is not invariant, since the Lorentz transformations transform any "static field" into "non-static" one.

1.2.4 Wave equations and wave solutions

From physical point of view when we talk about *waves* we mean *propagation of some disturbance, or perturbation, in a given medium*. It is also assumed that the perturbation does not alter the characteristic properties of the medium, and the time-evolution of the perturbation depends on the medium properties as well as on the specificities of the very perturbation. The waves are divided to 2 classes: *elementary* (*linear*) and *intrinsically coordinated* (*nonlinear*). The elementary waves are observed in homogeneous media and are generated by perturbing the equilibrium state of the medium through small quantities of external energy and momentum. The important properties of linear waves come from the condition, that during the propagation of the initial disturbance throughout the medium the structure of the medium does not change irreversibly, and the various such propagating perturbations do not interact with each other substantially. From mathematical point of view this means that the evolutionary equations, which are partial differential equations, describing such phenomena, are linear, so any linear combination with constant coefficients of solutions gives again a solution. In other words, the set of solutions of such equations is a real (finite or infinite dimensional) vector space.

The intrinsically coordinated, or nonlinear, waves disturb more deeply the medium structure, but the corresponding changes of the medium structure stay reversible. When subject to several such perturbations, the medium responses to the various disturbances is different in general, so the medium reorganization requires more complicated intrinsic coordination. All this demonstrates itself in various ways, depending on the medium properties and the initial perturbation. What we observe from outside is, that some important properties of the initial perturbations are changed in result of the interaction. In some cases we observe a time-stable coordination among the responding reactions of the medium and if the corresponding formation is finite, we may consider it as a new object. If this object keeps its energy and momentum we frequently call the corresponding medium *vacuum*. Clearly, such objects can exist only in appropriate media. In such cases, studying the objects, we get some information about the medium itself. From mathematical point of view these waves are described by nonlinear equations, so that a linear combination of solutions is not, as a rule, a new solution. The huge variety of various such cases could hardly be looked at from a single point of view, except some most general features.

It is important to note, that in the both cases, linear and nonlinear, the perturbations are bearable for the medium in the sense, that they do not destroy it. We are not going to consider here unbearable perturbations.

One common for every kind of waves characteristic is the *polarization*. The polarization determines the relation between the direction of propagation (at some point of the medium) and the direction of deviation from the equilibrium state of the medium point considered. If these two directions are parallel we say that the polarization is longitudinal, and if these directions are orthogonal we say the polarization is transverse. In general the polarization depends on the space-time point, i.e. it is a local characteristic. When the wave passes through some region of the medium, the points inside this medium commit some displacements along some (usually closed) trajectories. If these trajectories are straight lines we say that the polarization is *linear*, if they are circles we say the polarization is circular, etc. It is important to note that the polarization is an intrinsic property of the wave, therefore it ia an important for the theory characteristic. In particular, the mathematical character of the object (scalar, tensor, spinor, differential form, etc.), describing the wave, depends substantially on it. If the wave is linear, and the corresponding equation admits solutions with various polarizations, then summing up solutions with appropriate polarizations we can obtain a solution with a beforehand defined polarization.

Other common characteristics of the waves are the *propagation velocity*, determining the energy transfer from point to point of the medium, and *the phase surface*, built of all points, being in the same state with respect to the equilibrium state at a given moment. We are going to make use of these characteristics further.

Let us consider now the Maxwell's equations in regions far from sources: $\rho = 0$. We have

$$\frac{1}{c}\frac{\partial E}{\partial t} = rotB,\ divB = 0,\ \frac{1}{c}\frac{\partial B}{\partial t} = -rotE,\ divE = 0.$$

From the first and the second equations we obtain

$$rot(rotB) - \frac{1}{c}\frac{\partial}{\partial t}rotE = grad(divB) - \Delta B + \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} = -\Delta B + \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} = 0.$$

From the third and the fourth equations we obtain

$$-\Delta E + \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0.$$

These are the well known *wave equations*, and we are going to consider some of their properties. First we note, that these equations are just *necessary conditions for every solution of the vacuum Maxwell's equations.* Therefore, they may have solutions, which do not satisfy Maxwell's equations. Second, every component of E and B satisfies the same equation and does not depend on the rest ones. Third, these are second order differential equations of hyperbolic type.

We are interested in the following: *Do the vacuum Maxwell's equations admit finite and time-stable solutions, so that such solutions to serve as models of real objects?*. The positive answer to this question would be a serious virtue from the point of view of their adequacy as model equations for an important class of real objects, while the negative answer would make us searching for new equations, having solutions with the desired properties. At the beginning of the last century (about 1818), i.e. more than 40 years before the appearance of Maxwell's equations this problem has been essentially solved by Poisson, and because of its importance we shall consider it in some more detail.

Let's denote by u any component of the vector fields E and B. Then u satisfies the wave equation. We are interested in the behaviour of u at $t > 0$, if at $t = 0$ the function u satisfies the initial conditions

$$u|_{t=0} = f(x, y, z), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x, y, z).$$

Further we assume that the functions $f(x, y, z)$ and $F(x, y, z)$ are finite, i.e. they are different from zero in some finite region $D \subset \mathcal{R}^3$, which corresponds to the above introduced concept of a real object. Besides, we assume also that f is continuously differentiable up to third order, and F is continuously differentiable up to the second order. Under these conditions Poisson proved that a unique solution $u(x, y, z; t)$ of the wave equation is defined, and it is expressed by the initial conditions f and F by the following formula:

$$u(x, y, z, t) = \frac{1}{4\pi c}\left\{\frac{\partial}{\partial t}\left[\int_{S_{ct}} \frac{f(P)}{r} d\sigma_r\right] + \int_{S_{ct}} \frac{F(P)}{r} d\sigma_r\right\}, \tag{1.30}$$

where P is a point on the sphere S centered at the point (x, y, z) and a radius $r = ct$, $d\sigma_r$ is the surface element on $S_{r=ct}$.

The above formula (1.30) shows the following. In order to get the solution at the point (x, y, z), being at an arbitrary position with respect to the region

D, where the initial condition, defined by the two functions f and F, is concentrated, it is necessary and sufficient to integrate these initial conditions over a sphere S, centered at (x, y, z) and having a radius of $r = ct$. Clearly, the solution will be different from zero only if the sphere $S_{r=ct}$ crosses the region D at the moment $t > 0$. Consequently, if $r_1 = ct_1$ is the shortest distance from (x, y, z) to D, and $r_2 = ct_2$ is the longest distance from (x, y, z) to D, then the solution will be different from zero only inside the interval (t_1, t_2).

From another point of view this means the following. The initially concentrated perturbation in the region D begins to expand radially , it comes to the point (x, y, z) at the moment $t > 0$, makes it "vibrate" (i.e. our devices show the availability of a field) during the time interval $\Delta t = t_2 - t_1$, after this the point goes back to its initial condition and our devices find no more the field. Through every point out of D there will pass a wave, and its forefront reaches the point (x, y, z) at the moment t_1 while its backfront leaves the same point at the moment t_2. Roughly speaking, the initial condition "blows up" radially and goes to infinity with the velocity of light.

This mathematical result shows that *every* finite nonstatic solution of Maxwell's equations in vacuum is time-unstable, so these equations *have no* smooth enough time-dependent solutions, which could be used as models of real objects. As for the static solutions, as it was mentioned earlier, they also can not describe real objects.

These explicit results state clearly, that if we want to describe 3-dimensional time-dependent soliton-like electromagnetic formations (or configurations) it is necessary to leave off Maxwell's equations and to look for new equations for E and B, or for $F_{\mu\nu}$.

In relativistic notations the vacuum Maxwell's equations

$$\mathbf{d}F = 0, \ \ \delta F = 0$$

naturally, require F to satisfy the equation

$$(\mathbf{d}\delta + \delta\mathbf{d})F = \Delta F = 0,$$

which in standard coordinates $(x^1, x^2, x^3, x^4) = (x, y, z, \xi = ct)$ gives the usual wave equations for the components $F_{\mu\nu}$:

$$g^{\alpha\beta}\frac{\partial^2 F_{\mu\nu}}{\partial x^\alpha \partial x^\beta} = \frac{1}{c^2}\frac{\partial^2 F_{\mu\nu}}{\partial t^2} - \frac{\partial^2 F_{\mu\nu}}{\partial x^2} - \frac{\partial^2 F_{\mu\nu}}{\partial y^2} - \frac{\partial^2 F_{\mu\nu}}{\partial z^2} = 0.$$

Of course, not every solution of $\Delta F = 0$ is a solution to Maxwell's equations.

1.2.5 Plane electromagnetic waves

The exact solutions of Maxwell's equations in the whole 3-space, known as *plane electromagnetic waves*, are interesting not as some models of really existing objects, but as an convenient way to introduce some important characteristics of a class of EM-fields. The standard (i.e. the most widely spread) way to define such a solution is the following: *there is a rectangular coordinate system* (x, y, z)*, in which this solution depends on one space variable only.* The time-dependence of the solution is determined by the equations. Right now we note, that such a solution, if it exists, will be *infinite*! In fact, if z is the only coordinate, on which the solution depends, even if the dependence on z is finite, i.e.*localized and without singularities*, with respect to the rest two coordinates (x, y) this solution is *constant*; even at $x \to \infty,\ y \to \infty$ the values of the components of E and B, or $F_{\mu\nu}$, do not change. This simply means that the initial condition occupies the whole 3-space, or its infinite subregion, with finite values for the components of E and B. Since the integral energy W of every solution does not depend on time, we calculate it, making use of the initial condition, and obtain

$$W = \frac{1}{8\pi}\int (E^2 + B^2)dxdydz = \infty. \tag{1.31}$$

Let us now see how the plane wave looks like in the corresponding coordinate system, where E and B depend on z and t only. Since the derivatives with respect to x and y will be zero, from the wave equations we get

$$E = \Big[E_1(ct + \varepsilon z), E_2(ct + \varepsilon z), E_3(ct + \varepsilon z)\Big],$$

$$B = \Big[B_1(ct + \varepsilon z), B_2(ct + \varepsilon z), B_3(ct + \varepsilon z)\Big],\ \varepsilon = \pm 1.$$

Now the equations $divE = 0$ and $divB = 0$ require $E_3 = const$ and $B_3 = const$. Let us put these constants equal to zero since we do not interest in constant solutions. The Maxwell's equations reduce to

$$\frac{1}{c}\frac{\partial B_1}{\partial t} = \frac{\partial E_2}{\partial z},\ \frac{1}{c}\frac{\partial E_2}{\partial t} = \frac{\partial B_1}{\partial z},$$

$$\frac{1}{c}\frac{\partial E_1}{\partial t} = -\frac{\partial B_2}{\partial z},\ -\frac{1}{c}\frac{\partial B_2}{\partial t} = \frac{\partial E_1}{\partial z}.$$

These equations have the following solution:

$$E = \Big[E_1(ct + \varepsilon z), E_2(ct + \varepsilon z), 0\Big] = \Big[u(ct + \varepsilon z), p(ct + \varepsilon z), 0\Big],$$

$$B = \Big[B_1(ct+\varepsilon z), B_2(ct+\varepsilon z), 0\Big] = \Big[\varepsilon p(ct+\varepsilon z), -\varepsilon u(ct+\varepsilon z), 0\Big].$$

For the Poynting's vector we obtain $4\pi S = \Big[0, 0, -\varepsilon c(u^2+p^2)\Big]$. This solution, obviously, has the properties

$$E.B = 0,\ E^2 - B^2 = 0,$$

i.e. *the field has zero invariants*. Now we show in relativistic terms how the requirement for zero invariants $I_1 = I_2 = 0$ determines the solution *plane wave*. According to subsection **1.1.2** at zero invariants the energy-momentum tensor $Q_{\mu\nu}$, defined by (1.7), has only zero eigen values and unique isotropic eigen direction, defined by the couple of opposite isotropic vectors $\pm V = \varepsilon V$. Making use of the representation (1.13) for F and $*F$ we obtain for $Q_{\mu\nu}$

$$Q_\mu^\nu = -A_\sigma A^\sigma V_\mu V^\nu = -(A^*)_\sigma (A^*)^\sigma V_\mu V^\nu.$$

On the solutions of Maxwell's equations we shall have

$$0 = \nabla_\nu Q_\mu^\nu = -A^2 V^\sigma \nabla_\sigma V_\mu - V_\mu \nabla_\sigma (A^2 V^\sigma).$$

This relation shows, that the integral lines of the vector field V are *isitropic geodesics*, i.e. straight lines. Let's now choose the coordinates (x, y, z, ξ) in such way that the integral lines of V to lie entirely in the plane (z, ξ). Since $V_4 \neq 0$ always, we can suppose $V_4 = 1$. Then in these coordinates we shall have $V = (0, 0, \varepsilon, 1)$ and

$$F_{12} = F_{34} = 0,\ F_{13} = \varepsilon F_{14},\ F_{23} = \varepsilon F_{24},$$

$$A = (F_{14}, F_{24}, 0, 0),\ A^* = (-F_{23}, F_{13}, 0, 0) = (-\varepsilon A_2, \varepsilon A_1, 0, 0).$$

Clearly, in these notations A and A^* are the relativistic equivalents of E and B respectively.

Denoting $F_{14} = u$, $F_{24} = p$, for $\mathbf{d}F = 0$ and $\delta F = 0$ we get

$$\mathbf{d}F = \varepsilon(p_x - u_y)dx \wedge dy \wedge dz + (p_x - u_y)dx \wedge dy \wedge d\xi +$$

$$\varepsilon(u_\xi - \varepsilon u_z)dx \wedge dz \wedge d\xi + \varepsilon(p_\xi - \varepsilon p_z)dy \wedge dz \wedge d\xi = 0,$$

$$\delta F = (u_\xi - \varepsilon u_z)dx + (p_\xi - \varepsilon p_z)dy + \varepsilon(u_x + p_y)dz + (u_x + p_y)d\xi = 0.$$

From these equations it follows $u_x + p_y = 0$ and $u_y - p_x = 0$ and from these last relations we get the equations $u_{xx} + u_{yy} = 0$, $p_{xx} + p_{yy} = 0$, i.e. u

and p are harmonic functions with respect to the variables (x, y). Since we are searching for *finite* solutions in the *whole space*, from the well known properties of the harmonic functions it follows that u and p do not depend on $(x, y,)$. Thus,

$$u = u(\xi + \varepsilon z),\ p = p(\xi + \varepsilon z).$$

It is seen that the dependence of the field components on the unique space-variable z stays arbitrary, i.e. *can not be determined by the Maxwell's equations*. Obviously, if (k, φ_0) and (l, ψ_0) are two couples of real numbers, then $u(k\xi + \varepsilon kz + \varphi_0)$ and $p(l\xi + \varepsilon lz + \psi_0)$ define again a solution. On the other hand the numbers k, l define two constant isotropic vectors $\mathbf{k} = (0, 0, \varepsilon k_3, k_4),\ \mathbf{l} = (0, 0, \varepsilon l_3, l_4),\ k_3 = k_4 = k,\ l_3 = l_4 = l$, so we can write down $u(k\xi + \varepsilon kz + \varphi_0) = u(k_\mu x^\mu + \varphi_0);\ p(l\xi + \varepsilon lz + \psi_0) = p(l_\mu x^\mu + \psi_0)$. The vectors $(\mathbf{k}, k), (\mathbf{l}, l)$, or just their space-like parts $\mathbf{k}$ and $\mathbf{l}$ are called *wave vectors* of the two independent solutions

$$F(u) = \varepsilon u dx \wedge dz + u dx \wedge d\xi,\ F(p) = \varepsilon p dy \wedge dz + p dy \wedge d\xi,$$

and the quantities φ_0, ψ_0 are called *phase constants*. The quantities $(k_\mu x^\mu + \varphi_0),\ (l_\mu x^\mu + \psi_0)$ are called *phases*. Every of the two solutions is called *linearly polarized plane wave*.

It seems important to note that *the general plane wave is a sum, or a linear combination, of two entirely independent (except the common direction of propagation) linearly polarized plane waves*. Because of the linearity of Maxwell's equations a sum of linearly polarized plane waves with different directions of propagation is also a solution, but it is no more a plane wave. To propagate as a whole along a definite direction is one of the specific properties of the plane waves but this is not enough to use it as a model solution for real objects because of their infinity: infinite volume, infinite energy, infinite momentum and angular momentum.

A special interest for the theory is the choice of the two functions u and p as elementary periodic functions, namely

$$u = U_0 cos(k_\mu x^\mu + \varphi_0),\ p = P_0 cos(l_\mu x^\mu + \psi_0),$$

since most of the real EM-fields show definite properties of *periodicity*. In such case the quantities k_4/c and l_4/c are usually called *frequency* ν, (or *circular frequency* $\omega = 2\pi\nu$), and the quantity $\lambda = |\mathbf{k}|^{-1}$ is called *wave length*. So we can write down (e.g. for u)

$$\frac{1}{|\mathbf{k}|} = \lambda = \frac{c}{\nu} = cT,$$

where $T = 1/\nu$ is called *period*. Such waves are usually called *harmonic*. We'd like to note specially, that this *time-periodicity is a consequence of the specially chosen initial condition, namely that the function u, considered as a function of one variable, is a periodic function of the space-variable z*. This periodicity is *admissible* by the Maxwell's equations, but it is not a necessary consequence of these equations. So, the introduced "wave" characteristics come from a special class of initial conditions and nothing more. Finally we note that the general plane wave is determined by 4 real parameters k, l, φ_0, ψ_0 and 2 arbitrary functions of one (and the same) independent variable.

Two linearly polarized harmonic plane waves of the kind

$$E_1 = \Big[U_0 cos(\nu t + \varepsilon\frac{z}{\lambda} + \varphi_0),\ 0,\ 0\Big],\ B_1 = \Big[0,\ -\varepsilon U_0 cos(\nu t + \varepsilon\frac{z}{\lambda} + \varphi_0),\ 0\Big];$$

$$E_2 = \Big[0,\ P_0 sin(\nu t + \varepsilon\frac{z}{\lambda} + \varphi_0),\ 0\Big],\ B_2 = \Big[\varepsilon P_0 sin(\nu t + \varepsilon\frac{z}{\lambda} + \varphi_0),\ 0,\ 0\Big]$$

are called *consistent*. Summing them up we obtain again a harmonic plane wave

$$E = \Big[U_0 cos(\nu t + \varepsilon\frac{z}{\lambda} + \varphi_0),\ P_0 sin(\nu t + \varepsilon\frac{z}{\lambda} + \varphi_0),\ 0\Big],$$

$$B = \Big[\varepsilon P_0 sin(\nu t + \varepsilon\frac{z}{\lambda} + \varphi_0),\ -\varepsilon U_0 cos(\nu t + \varepsilon\frac{z}{\lambda} + \varphi_0),\ 0\Big].$$

We see that the consistent linearly cycling E and B of the two harmonic and linearly polarized waves create an illusion for circulating E and B of their superposition, i.e. the couple of orthogonal vectors (E, B) takes part in two motions: *advancing* along z and *circulating* in the plane (x, y) left-wise or right-wise with the frequency of ν. In this way we get an impression about an elliptically (or circularly at $U_0 = P_0$) polarized plane wave.

Note that the harmonic plane wave occupies the whole 3-space, its energy-density is the constant quantity $w \sim (U^2 + P^2)$, its full energy is infinite since the 3-dimensional integral of a constant over the whole space $\mathcal{R}^3$ is infinity.

Finally we note the transverse character of any plane wave, which is seen from the transverse direction of E and B with respect to the direction of propagation.

1.2.6 Potentials

According to the general notion of potential this is a scalar or some tensor field, from which by differentiating (once or more) the *physical* field, i.e. the

field that is physically measured, is obtained. The most frequent case is a "single" differentiating, but there are examples of "double" differentiating. Such is the case of gravitational field in the frame of General Relativity, where the curvature field, which is identified with the physical field, is obtained from the metric tensor through double differentiating. As we shall see now, some of the solutions of Maxwell's equations can also be obtained by double differentiating of some of the solutions of the wave equation. This is the method of Hertz potentials. In fact, if A_1 and A_2 are two solutions (vector fields) of the vector wave equation $\Box A = 0$, then the expressions

$$E = rot(rotA_1) - \frac{\partial}{\partial\xi}rotA_2,\ B = rot(rotA_2) + \frac{\partial}{\partial\xi}rotA_1$$

define a solution to Maxwell's equations. In relativistic notations we have

$$(\Box G)_{\mu\nu} = -\Big[(\mathbf{d}\delta + \delta\mathbf{d})G\Big]_{\mu\nu} = 0.$$

Then

$$F = \mathbf{d}\delta G = -\delta\mathbf{d}G$$

is a solution to Maxwell's equations $\delta F = 0,\ \mathbf{d}F = 0$. In fact, since $\mathbf{d}\circ\mathbf{d} = 0$, $\delta\circ\delta = 0$ and $\Box\circ\mathbf{d} = \mathbf{d}\circ\Box,\ \Box\circ\delta = \delta\circ\Box$, then

$$\delta F = \delta(\mathbf{d}\delta G) = (\Box - \mathbf{d}\delta)\delta G = \delta\Box G = 0,\ \mathbf{d}F = \mathbf{d}(\mathbf{d}\delta G) = 0.$$

These solutions are used for description of the Hertz vibrator's radiation. The spherical wave

$$(A_1, A_2, A_3) = \left[\frac{a_1(ct-r)}{r},\ \frac{a_2(ct-r)}{r},\ \frac{a_3(ct-r)}{r}\right],$$

which is a solution of the wave equation, but *is not* a solution to the Maxwell's equations, is used in the above shown way to build a solution of Maxwell's equations. Clearly, the standard choice of the components a_i as elementary harmonic functions brings a singularity at the point $x = y = z = 0$.

The general solution of Maxwell's equations with non-zero and *not depending* on the field electric current, is obtained by means of introducing the 4-potential, i.e. an 1-form $A = A_\mu dx^\mu$ and defining $F = \mathbf{d}A$, which is *always* possible, since in CED we have in all cases $\mathbf{d}F = 0$. The 1-form A is

determined up to a term of the kind $\mathbf{d}f$, since A and $A+\mathbf{d}f$ give the same F. Then

$$\delta F = \delta\mathbf{d}A = (\Box - \mathbf{d}\delta)A = \Box(A+\mathbf{d}f) - \mathbf{d}\delta(A+\mathbf{d}f).$$

We see that choosing f as a solution to the equation $-\delta\mathbf{d}f = \Box f = \delta A$ we can redefine A as $A' = A+\mathbf{d}f$. Clearly A and A' define the same F, and $\delta A' = 0$. So, the 1-form A' satisfies the equation $\delta\mathbf{d}A' = \Box A' = 4\pi j$, the solutions of which are well known when the current j *does not depend* on A'. Of course, when the change in the mechanical energy of the charge-carriers is taken into account, and these carriers do not "appear" and "disappear", then according to subsection (1.1.2), equation (1.9), the 4-current $j_\mu = \rho u_\mu$ depends on the field F, and the equations become nonlinear. Therefore, it is an illusion to think, that the 4-potential approach solves the problem completely: in all really significant cases the 4-current depends substantially on the field F in accordance with equation (1.9), which, in turn, follows from the energy-momentum conservation law and could hardly be put into some doubt. This fact shows some inadequacy of the thesis for the universal character of the 4-potential approach, its (always possible) introduction does not lead to a complete solution of the entire problem. In all cases the energy-momentum conservation requires nonlinear inter-relations between the current and the field.

1.3 Amplitude and Phase

1.3.1 Amplitude and phase of a plane wave

The importance of the concepts of *amplitude* and *phase* in the electromagnetic theory is out of any doubt, but sufficiently general and universal definitions of these concepts in CED are still missing. Our purpose in this section is to consider some new ways to introduce these concepts into theory through a pure algebraic and coordinate free approach in both, nonrelativistic and relativistic formalisms. We consider first the case of a plane wave solution.

In the corresponding coordinate system the plane wave solution is

$$F = \varepsilon U_0 cos(k_\mu x^\mu + \varphi_0)dx\wedge dz + U_0 cos(k_\mu x^\mu + \varphi_0)dx\wedge d\xi +$$

$$\varepsilon P_0 sin(l_\mu x^\mu + \psi_0)dy\wedge dz + P_0 sin(l_\mu x^\mu + \psi_0)dy\wedge d\xi,$$

or, in terms of E and B

$$E = \Big[U_0 cos(k_\mu x^\mu + \varphi_0),\ P_0 sin(l_\mu x^\mu + \psi_0),\ 0\Big],$$

$$B = \Big[\varepsilon P_0 sin(l_\mu x^\mu + \psi_0), -\varepsilon\ U_0 cos(k_\mu x^\mu + \varphi_0),\ 0\Big].$$

As we mentioned, the quantity $k_\mu x^\mu + \varphi_0 = \nu t + \varepsilon z/\lambda + \varphi_0$ is called *phase* of the plane wave. As for the amplitude, according to the usual sense of this concept, it is the *magnitude*, or the *maximum value* of a given quantity. In our case we have a couple (E, B) of vector fields, so it seems natural to define the amplitude by the relation

$$\sqrt{E^2 + B^2} = \sqrt{U_0^2 + P_0^2},$$

i.e. a square root of the energy-density. For the vector product $E \times B$ we obtain

$$E \times B = \Big[(0, 0, -\varepsilon(U_0^2 + P_0^2)\Big].$$

This is a constant vector.

Now, the triple $(E, B, E \times B)$ defines a basis of the tangent (or cotangent) space at every point, where the field is different from zero (we assume $E \neq 0, B \neq 0$). Moreover this is an orthogonal basis. We denote this basis by $\mathcal{R}_1$, so we can write $\mathcal{R}_1 = (E, -\varepsilon B, -\varepsilon E \times B)$. From the properties of the plane wave solutions we obtain $|E| = |B|$. But, the physical dimension of the third vector $E \times B$ is different from that of the first two. So, we introduce the factor α

$$\alpha = \frac{1}{\sqrt{\frac{E^2+B^2}{2}}}.$$

Making use of α, we introduce the basis

$$\mathcal{R} = \Big[\alpha E, -\varepsilon\alpha B, -\varepsilon\alpha^2 E \times B\Big].$$

Hence, at every point we've got two bases: $\mathcal{R}$ and the coordinate basis $\mathcal{R}_0 = \Big[\partial_x, \partial_y, \partial_z\Big]$, as well as the corresponding co-bases $\mathcal{R}^*$ and $\mathcal{R}_0^* = (dx, dy, dz)$. We are interesting in the invariants of the corresponding transformation matrix $\mathcal{M}$ from $\mathcal{R}_0^*$ to $\mathcal{R}^*$. It is defined by the relation $\mathcal{R}_0^*.\mathcal{M} = \mathcal{R}^*$. So, we obtain

$$\mathcal{M} = \begin{Vmatrix} \alpha E_1 & -\varepsilon\alpha B_1 & -\varepsilon\alpha^2(E \times B)_1 \\ \alpha E_2 & -\varepsilon\alpha B_2 & -\varepsilon\alpha^2(E \times B)_2 \\ \alpha E_3 & -\varepsilon\alpha B_3 & -\varepsilon\alpha^2(E \times B)_3 \end{Vmatrix}.$$

We shall try to express the amplitude and the phase of the plane wave as functions of the invariants of this matrix. So, in all cases , where this is possible, the invariant character of the so defined phase and amplitude will be out of doubt. As it is well known, in general, every square $n \times n$-matrix $\mathcal{L}$ has n invariants $I_1, I_2, ..., I_n$, where I_k is the sum of all principle minors of order k. The invariant $I_1(\mathcal{L}) = \mathcal{L}_{11} + ... + \mathcal{L}_{nn}$ is the sum of all elements on the principle diagonal, and the invariant $I_n = det(\mathcal{L})$ is the determinant of the matrix. In our case $n = 3$, so for the invariant I_2 we get

$$I_2 = \det \begin{Vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{Vmatrix} + \det \begin{Vmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{Vmatrix} + \det \begin{Vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{Vmatrix}.$$

We compute $I_3(\mathcal{R})$.

$$I_3(\mathcal{R}) = \det \begin{Vmatrix} \alpha E_1 & -\varepsilon\alpha B_1 & -\varepsilon\alpha^2(E \times B)_1 \\ \alpha E_2 & -\varepsilon\alpha B_2 & -\varepsilon\alpha^2(E \times B)_2 \\ \alpha E_3 & -\varepsilon\alpha B_3 & -\varepsilon\alpha^2(E \times B)_3 \end{Vmatrix} = \alpha^4(E \times B)^2.$$

Now the amplitude $\mathcal{A}$ of the plane electromagnetic wave may be defined as follows:

$$\mathcal{A} = \sqrt[2]{\alpha^{-2} I_3(\mathcal{R})} = \sqrt[2]{\alpha^2(E \times B)^2} = \alpha |E \times B|.$$

1.3.2 Amplitude and phase of a general field

The invariant character of the above given definition of the plane wave amplitude suggests its natural extending to an arbitrary field. So, if the couple (E, B) represents the field, we introduce the matrix $\mathcal{M}(\mathcal{R})$ of the basis $\mathcal{R} = \Big[\alpha E, -\alpha B, -\alpha^2(E \times B)\Big]$ and define the amplitude $\mathcal{M}$ of the field by

$$\mathcal{A}(E, B) = \sqrt[2]{\alpha^{-2} I_3(\mathcal{R})} = \alpha |E \times B|. \tag{1.32}$$

We go further now to define the phase in the general case. We'll make use of the matrix of the basis

$$\mathcal{R} = \Big[\alpha E, -\alpha B, -\alpha^2 E \times B\Big],$$

defined by the general field (E, B). The invariants

$$I_1(\mathcal{R}) = \alpha E_1 - \alpha B_2 - \alpha^2(E \times B)_3,$$

$$I_2(\mathcal{R}) = -\alpha^2(E\times B)_3 + \alpha^3\Big[E(E.B) - B(E.E)\Big]_2 + \alpha^3\Big[E(B.B) - B(E.B)\Big]_1,$$

$$I_3(\mathcal{R}) = \alpha^4(E\times B)^2,$$

obviously, are physically dimensionless. When the inequality

$$\frac{1}{2}\Big|I_1(\mathcal{R}) - 1\Big| \leq 1,$$

holds, then the function *arccos* is defined on the expression on the left. In these cases, by definition, the phase φ of the field (E, B) shall be defined by

$$\varphi = arccos\left[\frac{1}{2}\Big[I_1(\mathcal{R}) - 1\Big]\right] \tag{1.33}$$

For the plane wave solution

$$E = \Big[u(ct+\varepsilon z), p(ct+\varepsilon z), 0\Big],\ B = \Big[p(ct+\varepsilon z), -u(ct+\varepsilon z), 0\Big]$$

we get

$$I_1(\mathcal{R}) = I_2(\mathcal{R}) = \frac{2u}{\sqrt{u^2+p^2}} + 1 = \frac{2E_1}{|E|} + 1,$$

and for a circularly polarized plane monochromatic wave we get $\varphi = k_\mu x^\mu + const.$

Let's now see when the basis $\mathcal{R}$ is normed, i.e. when

$$|\alpha E| = 1,\ |\alpha B| = 1,\ \alpha^2|E\times B| = 1.$$

From the first two equations it follows $|E| = |B|$, and from the third equation it follows $E.B = 0$. Moreover, the relations $|E| = |B|,\ E.B = 0$ follow from the third equation only: $\alpha^2|E\times B| = 1$. So, the normed character of $\mathcal{R}$ leads to its orthonormal character, consequently, $\det\mathcal{M}(\mathcal{R}) = 1$. Vice versa, the requirement $\det\mathcal{M}(\mathcal{R}) = 1$ leads to the orthonormal character of $\mathcal{R}$. We obtain, that the requirement $\det\mathcal{M}(\mathcal{R}) = 1$ is equivalent to the null character of the field: $I_1 = B^2 - E^2 = 0,\ E.B = 0$.

The relations obtained suggest to define and consider the following 4-linear map:

$$R(x,y,v,w) = \det\begin{Vmatrix} x_1 & y_1 & (v\times w)_1 \\ x_2 & y_2 & (v\times w)_2 \\ x_3 & y_3 & (v\times w)_3 \end{Vmatrix}.$$

The following relations hold:

$$R(x,y,v,w) = (x\times y).(v\times w) = \Big[y\times(v\times w)\Big].x = \Big[(v\times w)\times x\Big].y,$$

$$R(x,y,v,w) = -R(y,x,v,w),$$

$$R(x,y,v,w) = -R(x,y,w,v),$$

$$R(x,y,v,w) + R(x,v,w,y) + R(x,w,y,v) = 0,$$

$$R(x,y,v,w) = R(v,w,x,y),$$

$$R(x,y,x,y) = (x\times y)^2.$$

We note that this 4-linear map has all algebraic properties of the Riemannian curvature tensor, therefore in the frame of this section, we shall call it *algebraic curvature*. For the corresponding 2-dimensional curvature $K(x,y)$, determined by the two vectors (x,y) we obtain

$$K(x.y) = \frac{R(x,y,x,y)}{x^2y^2-(x.y)^2} = \frac{(x\times y)^2}{x^2y^2(1-cos^2(x,y))} = \frac{x^2y^2sin^2(x,y)}{x^2y^2sin^2(x,y)} = 1.$$

Let (e_1,e_2,e_3) be a basis. We compute the corresponding Ricci tensor R_{ik} and the scalar curvature $\mathbf{R}$.

$$R_{ijkl} = R(e_i,e_j,e_k,e_l) = (e_i\times e_j).(e_k\times e_l),$$

$$R_{ik} = \sum_l R^l_{ikl} = (e_i\times e_1).(e_k\times e_1) + (e_i\times e_2).(e_k\times e_2) + (e_i\times e_3).(e_k\times e_3),$$

$$\mathbf{R} = \sum_i R^i_i = 2\Big[(e_1\times e_2)^2 + (e_1\times e_3)^2 + (e_2\times e_3)^2\Big].$$

For our basis $\mathcal{R}_1$ we obtain the following non-zero components:

$$R_{12,12} = 4\frac{E^2.B^2}{(E^2+B^2)^2}sin^2(E,B),$$

$$R_{13,13} = R_{12,12}.\frac{2E^2}{E^2+B^2},\ \ R_{23,23} = R_{12,12}.\frac{2B^2}{E^2+B^2},$$

and for the scalar curvature we get

$$\mathbf{R}(E,B) = 24\frac{E^2B^2}{(E^2+B^2)^2}sin^2(E,B).$$

After this short retreat let's go back to the quantities *phase* and *amplitude*. The above mathematical consideration suggests to try to relate these two concepts with the notion of curvature in purely formal sense, namely as a 2-form with values in the bundle $L_{T(\mathcal{R}^3)}$ of linear maps in the tangent bundle. Most generally, a 2-form R with values in the bundle $L_{T(\mathcal{R}^3)}$ looks as follows

$$R = \frac{1}{2}R^k_{ijl}dx^i \wedge dx^j \otimes \frac{\partial}{\partial x^k} \otimes dx^l.$$

We have to determine the coefficients R^k_{ijl}, i.e. we have to define a (3×3)-matrix $\mathcal{R}$ of 2-forms. We define this matrix in the following way:

$$\mathcal{R} = \begin{Vmatrix} \alpha E_1 dy \wedge dz & \alpha B_1 dy \wedge dz & \alpha^2(E \times B)_1 dy \wedge dz \\ \alpha E_2 dz \wedge dx & \alpha B_2 dz \wedge dx & \alpha^2(E \times B)_2 dz \wedge dx \\ \alpha E_3 dx \wedge dy & \alpha B_3 dx \wedge dy & \alpha^2(E \times B)_3 dx \wedge dy \end{Vmatrix}.$$

The columns of this matrix are the 2-forms $*E,\ *B,\ *(E \times B)$, multiplied by the factor α at some degree in order to obtain physically dimensionless quantities.

We aim to define the amplitude and the phase of the field (E, B), making use of this matrix. The amplitude $\mathcal{R}$ of the field we define by

$$\mathcal{A} = \frac{1}{3\alpha}R_{ijkl}R^{ijkl} = \frac{2}{3\alpha}\left[1 + 2\frac{(E \times B)^2}{(E^2 + B^2)^2}\right]. \tag{1.34}$$

In order to define the phase we first consider the 2-form $tr \circ R$. We get

$$tr \circ \mathcal{R} = \alpha E_1 dy \wedge dz + \alpha B_2 dz \wedge dx + \alpha^2(E \times B)_3 dx \wedge dy.$$

The square of this 2-form is

$$(tr \circ \mathcal{R})^2 = \alpha^2\left[(E_1)^2 + (B_2)^2 + \alpha^2[(E \times B)_3]^2\right].$$

Now the phase φ of the field we define by

$$\varphi = arccos\left[\sqrt{\left|\frac{(tr \circ R)^2 - 1}{2}\right|}\right]. \tag{1.35}$$

whenever the right-hand expression is well defined.

Let's compute this quantity for a plane wave, moving along the z-coordinate from $-\infty$ to $+\infty$:

$$E = \Big[u(z-ct),\ p(z-ct),\ 0\Big],\ B = \Big[-p(z-ct),\ u(z-ct),\ 0\Big].$$

We get

$$(E \times B) = \Big[0,\ 0,\ (u^2+p^2)\Big],\ \alpha = \frac{1}{\sqrt{u^2+p^2}},$$

$$tr\circ R = \frac{u}{\sqrt{u^2+p^2}}dy\wedge dz + \frac{u}{\sqrt{u^2+p^2}}dz\wedge dx + dx\wedge dy,\ (tr\circ R)^2 = 1 + \frac{2u^2}{u^2+p^2},$$

$$\mathcal{A} = \sqrt{u^2+p^2},\ \varphi = arccos\left(\frac{|u|}{\sqrt{u^2+p^2}}\right)$$

In the case of a plane harmonic wave $\varphi = k_\mu x^\mu + \varphi_0$. Note that since in this coordinate system the components of the plane wave do not depend on the coordinates x and y, the corresponding 2-form $tr\circ R$ is *closed*: $\mathbf{d}(tr\circ R) = 0$.

If we work in relativistic terms, it is necessary to introduce some changes in the matrix of 2-forms. First, we add one more column and one more row. Second, instead of $*E,\ *B,\ *(E\times B)$ it is more convenient to use their dual with respect to the pseudoeucledean $*$-operator $*_4(*E),\ *_4(*B),\ *_4(*E\times B)$. So, the matrix $\mathcal{R}$ takes the form

$$\begin{Vmatrix} \alpha E_1 dx\wedge d\xi & \alpha B_1 dx\wedge d\xi & \alpha^2(E\times B)_1 dx\wedge d\xi & \alpha^2(E\times B)_1 dx\wedge d\xi \\ \alpha E_2 dy\wedge d\xi & \alpha B_2 dy\wedge d\xi & \alpha^2(E\times B)_2 dy\wedge d\xi & \alpha^2(E\times B)_2 dy\wedge d\xi \\ \alpha E_3 dz\wedge d\xi & \alpha B_3 dz\wedge d\xi & \alpha^2(E\times B)_3 dz\wedge d\xi & \alpha^2(E\times B)_3 dz\wedge d\xi \\ 0 & 0 & 0 & 0 \end{Vmatrix}.$$

Respectively, we obtain

$$tr\circ R = \alpha E_1 dx\wedge d\xi + \alpha B_2 dy\wedge d\xi + \alpha^2(E\times B)_3 dz\wedge d\xi,$$

$$\frac{1}{2}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = -2,$$

$$(tr\circ R)^2 = -\alpha^2\Big[(E_1)^2 + (B_2)^2 + \alpha^2[(E\times B)_3]^2\Big].$$

In these terms the definitions for the *amplitude* $\mathcal{A}$ and the *phase* φ will look as follows

$$\mathcal{A} = -\frac{1}{4\alpha} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}, \ \varphi = arccos\left[\sqrt{\left|\frac{|(tr \circ R)^2| - 1}{2}\right|}\,\right]. \tag{1.36}$$

For the solution plane wave the 2-form $tr \circ R$ is

$$tr \circ R = \frac{u}{\sqrt{u^2+p^2}} dx \wedge d\xi + \frac{u}{\sqrt{u^2+p^2}} dy \wedge d\xi + dz \wedge d\xi$$

and in the general case it is not closed. Since $cos\varphi = u/\sqrt{u^2+p^2}$, then the equation $\mathbf{d}(tr \circ R) = 0$ requires (in this coordinate system) the following conditions to be fulfilled: $\varphi_x = \varphi_y, \ \varphi_z = 0$.

With these remarks on the amplitude and phase of the EM-field in CED we come to a close of our short review of the basic principles, concepts and relations in Maxwell's theory. As it is clear from what we presented so far our purpose is not to describe positive prescriptions for getting results in this theory. We have tried to pay attention to those moments of the theory, which show directly or indirectly, its frame of applicability. Doing this, we get a clearer notion of how and what to alter in view of what kind of objects we want to describe. In the next section we summarize those points of of Maxwell's theory in order to have clearer and more definite motivation for the appropriate to our aims changes in the theory.

1.4 Why and What to Change in Classical Electrodynamics

If the question *why do we want to change something in CED* is raised, we respond shortly in the following way: *because we want to enrich CED with new areas of applicability, extending in a natural way the class of admissible solutions, aiming to describe (3+1)-dimensional soliton-like objects in the pure field case, as well as in the presence of active external fields (media).*

At the end of the last and the beginning of this century it has become clear that some experimentally established facts can not be understood and explained in the frame of Faraday-Maxwell's electrodynamics. For example,

1. Why the initiation of photochemical reactions depends on the color and not on the intensity of light?

2. Why the velocities of the photoelectrons does not depend on the intensity of light?

3. Why the shortwave radiation is emitted from bodies at high temperature?

4. Why the shortwave radiation is chemically more active than the longwave one?

More generally: *why the influence of light on matter depends qualitatively on its color and not on its intensity*?

These and other experimentally established facts motivate a serious analysis of Maxwell's electrodynamics. In result, Planck and, later, Einstein set fort the discrete point of view on the structure of the electromagnetic field. The later experiments of Compton proved the truth of this viewpoint and the notion of *photon* as an elementary electromagnetic formation (natural object) was created. Soon the photons were provided with integral characteristics like *frequency, energy, momentum, spin*. The Planck constant h turned into an omnipresent parameter, serving to separate the real photons from other theoretically admissible electromagnetic formations. The Planck's formula $E = h\nu$ is the essential criterion for reality of the objects considered. This formula clearly says that only those elementary electromagnetic formations can really exist, the existence of which is strongly bound up with the availability of an intrinsic periodic process with a period of $T = 1/\nu$, and the corresponding to this process integral action $E.T$ is exactly equal to the Planck constant: $h = E.T$. May be this limitation seems to be very strong, but we have no reasons not to trust it so far. Anyway, it is clear that the Planck formula separates a class of natural objects being characterized by the elementary action of h. Moreover, this intrinsic periodic process and the fact, that every photon moves as a whole uniformly by the same velocity c, no matter what its frequency is, support the notion that they are finite soliton-like objects. Otherwise it is hardly understandable where the quantity *frequency* can come from as a characteristic of a *free and uniformly moving point-like, i.e. structureless, object*.

These and other circumstances set a clear chalenge before the those days theoretical physics: description of 3-dimensional, finite soliton-like objects, having all integral properties of the free photons. The established through time quantum-probabilistic approach does not solve this problem since it is built on other principles and purcues other objectives. Modern quantum

electrodynamics, although its serious achievements in describing the atomic phenomena, also works with the assumption "structureless photon" and is interested mainly in its integral propeties.

In view of our intention to build a soliton-like model of the free photon a serious analysis of the initial and basic principles of classical electrodynamics had to be made. The purpose of this ananlysis is to find what to preserve and what to change, i.e. to find those points of the Faraday-Maxwell theory, the appropriate change of which will be of use and will not bring us to any undesirable complications. Being fully aware of the significance of Faraday-Maxwell theory in physics and in all our knowledge of the natural world, we present our conclusions fairly and in a transperant way as far as we are able to do this. As we said earlier, in doing this we follow the rule that *the respect and esteem paid to the creators can not be honest and genuine if they are not in correspondence with the respect and esteem paid to the Truth.*

1. The elementary spherically symmetric and topologically nontrivial solution (see 1.2.2) for the electric field E ($B = 0$), which could be identified with the only spherically symmetric representative of the corresponding cohomology class, defines in fact the electric charge as a topological invariant. The important point here, we'd like to mention, is the static character of the solution-field obtained, so where a test particle, placed somewhere in the field, should take *momentum* from in order to change its own momentum according to the momentum conservation law, i.e. the equations of motion, is not quite clear. Since $B = 0$, the Poynting vector $S = E \times B$, traditionally considered as describing the local momentum-transport of the field, is equal to zero. Clearly, the static character of the field is an illusive feature and does not give an adequate picture of a real situation. But it is an *exact*, so a trustworthy, solution! Because of the radial direction of the field strength, i.e. of the particles' momentum change, it seems hardly possible to avoid the notion, that some *real objects move radially and carry momentum to and from the source*. Since there is no momentum accumulation at the source object the same momentum has to be carried "to" and "from" for a unit time. As for the intrinsic mechanism of momentum exchange between the field and the particle CED tells nothing, it gives just the final integral effect. For the general static case, according to (1.2.3), because of the well known properties of the solutions to Laplace equation, they are not suitable for models of real objects. So, our general conclusion is that *no static solution of Maxwell's equations presents a sufficiently adequate picture of real objects*

and processes .

2. The computation of the full interaction energy for two spherically symmetric fields in (1.2.3) clearly shows, that while the interaction of the two fields is a local fact and takes place at every point where the two fields are different from zero, the interaction energy is an integral characteristic and in no case should be identified with some potential. From this point of view the Coulomb force, describing this interaction as a change of the mechanical momentum of the charge-carriers, can not be a local characteristic, since it takes into account the contribution to this process of the interaction at all points. The "close" form of $*\omega$ and dW, where W is considered as a function of the points, where the two charges stay, is not a sufficient ground the *integral* characteristic "interaction force" to be identified with $*\omega$. As for the potential U, introduced by the relation $dU = *\omega$, it is a local characteristic too, so $dU \neq dW$ always.

3. The considerations in (1.2.4) are of basic significance for us: every localized finite initial condition determines unique solution of the wave equation, the future time-behaviour of which could shortly be characterized as a "radial blow-up". So, the same is true for the pure field Maxwell equations. An important feature of the 3-dimensional case is the availability of "forefront" and "backfront", which simply means, that every point of the space will "feel" the field a finite period of time, after which it will "forget" what happened. This result leaves no chance and hopes for making use of Maxwell equations to obtain soliton-like solutions, they have no such solutions in the whole space. Although undesirable, this conclusion is unavoidable. This is the mathematical reality and we have to accept it with the corresponding respect.

4. As we noted in (1.2.5) the popular and well liked solution of the pure field Maxwell equations *plane wave* is *unphysical, unrealistic*, since it is *infinite: it possesses infinite integral energy and occupies an infinite 3-volume*. Besides, all periodic-wave solutions are defined by appropriate initial conditions, so the wave characteristics like *period, frequency* and *wave vector* come from these initial conditions, i.e. they are admissible by the equations but are not necessary for all solutions. So, other kind of solutions, having no such characteristics, are also admissible. But, electromagnetic radiation without such characteristics, has never been observed and is hardly possible. If this is

true, what to do with such solutions in view of the desired adequacy between the theory and the reality. Note that the representation of the plane wave $u(z - ct)$ through simple monochromatic plane waves (the so called wave packets) does not solve the problem since these packets are not time-stable objects.

5. The 4-potential approach is not of interest from our point of view, since for the pure field case it reduces the problem to a subclass of solutions of the wave equation for the components A_μ of this 4-potential, namely those, satisfying the additional condition $\nabla_\mu A^\mu = 0$, and so soliton-like solutions are excluded. In the presence of a current the problems with the nonlinear dependence of the current on the field, mentioned in (1.2.6), appear and are hardly avoidable in general. From our viewpoint the 4-potential approach is important rather to legalize the gauge fields in theoretical fields, although at an elementary (linear) abelean level $U(1)$. Unfortunately, even for nonabelean gauge fields, leading to the nonlinear Yang-Mills equations, (3+1)-soliton-like solutions are not found, moreover, there are some studies, showing that in the pure field case there are no such solutions. Let's not forget, that the so called *instantons*, i.e. solutions to the equations $*F = F$, are possible only at *positively defined* space-time metric and *zero energy-momentum tensor*, so they could be hardly considered as models of real objects. As for the *monopoles*, they want additional field, interacting with the gauge field in a special way.

6. In the case of continuous media CED adds to the free current j^μ an additional current j_b^μ, called *bound*. The two new vectors P and $\mathcal{M}$ are introduced (see 1.1.3) in the same way as the vectors E and B are connected to the free current. So, the number of the unknown functions becomes greater than the number of the equations, which makes the things difficult to overcome as the historic development shows. The introduction of coefficients-material constants by means of series developments seems to be a very useful practical skill, but it can not serve as a promising theoretical idea. The inherited inertia in thinking that energy-momentum exchange with the field can occur *only if charge carrying particles are present* still bears upon our minds. The *hypothesis* that the Faraday induction law is universally (i.e. for *all media, including vacuum*) valid has turned almost into a dogma, which, by the way, forbids energy-momentum exchange through $*F$. Even if some brave investigators admit an energy-momentum exchange between the field and the

medium to occur through $*F$, they begin to talk about magnetic charges (in analogy with the electric case) having similar to the electric charges properties.

From our point of view the real and important moment is the very *energy-momentum exchange*, and how this exchange is realized is an additional problem, depending on the special case considered. So, *the most important step in our approach is to find an appropriate model equation for this exchange, because this is the universal characteristic property of every interaction in nature*. All further specializations depend on the case under consideration.

Hence, in view of what was said so far and in view of the purposes we set, we come to the following conclusion. The algebraic construction *a couple of vector fields* (E, B) on $\mathcal{R}^3$, or a *differential 2-form on the Minkowski space-time*, is in general adequate to the field as far as it reflects well enough its algebraic and general polarization properties. Maxwell's equations do not reflect adequately enough the local properties of the field, therefore not all solutions can represent satisfactory models of real objects. So, our choice is to preserve, though in an altered form, the basic algebraic picture of the field in relativistic terms, but we'll replace Maxwell's equations with new nonlinear equations, the physical sense of which is to define how the local energy-momentum exchange between the field and some other continuous physical object (medium or field) is carried out. The reason to turn to the local energy-momentum conservation laws reflects our point of view that they are *more hopeful* and *more universal*.

Let us outline in few words our notion of the objects we want to describe. As we mentioned in (1.2.1) they must be *extended*, but *finite* and *time-stable*. Besides, their existence must be strongly connected with an *internal and intrinsic dynamics, in particular - periodic process*. The characteristics of the internal dynamics must be in strong relation with the integral characteristics of the object. It is necessary to find invariant quantities, separating the really existing objects from all theoretically admissible. If some interaction processes take place, a transformation of these objects to each other or to new ones, obeying definite conservation laws, should be possible. Various internal structures at different levels, as well as creation of stable structures out of these objects, should be also possible. A stability with respect to external disturbances must be available, so that such a disturbance to result finaly in the behaviour of the object as a whole: *uniform motion* when there

are no external disturbances and *accelerated motion* in presence of a bearable external disturbance. From mathematical point of view this means that at any moment the functions, describing our object(s), shall be different from zero inside a finite *connected* 3-dimensional region with trivial or non-trivial topology, while the time-evolution should be determined by the dynamic field equations.

Passing to the formulation of the basic principles and their mathematical adequacies of what will be called further *Extended Electrodynamics* (EED), we are fully aware of, that at this initial stage of our investigation the following two things should be obeyed. First, from purely pragmatic point of view, it seems better to preserve as admissible all solutions to Maxwell's equations as exact solutions to the new equations and to use them whenever it is possible. Second, the local energy-momentum conservation laws will probably define "weak" equations, so additional conditions (initial data or some new equations), reflecting some new specific features of the objects under consideration, have to be imposed on the solutions. In such cases we shall make use of the considerations and conclusions in (1.2.1). We prefer to work in relativistic terms, since we consider this language more adequate to the physical situations we want to describe.

Finally, we assume the *general covariance principle*. i.e. the understanding that physical sense may have those concepts and statements, which do not depend on the local coordinates used. Accordingly, we'll aim at a coordinate-free formulation of the basic statements and equations in all cases when this is possible, paying no attention to the simple Minkowski space background used.

1.5 Extended Electrodynamics

1.5.1 Physical conception for the EM-field in EED

As it was mentioned, the mathematical models in CED of the real electromagnetic fields in vacuum are "infinite", or if they are finite, they are strongly time-unstable. These models are not consistent with a number of experimental facts. A deeper analysis resulted in the new conception for a discrete character of the field. This physical understanding of the field is the true foundation of EED and it shows clearly the principle differences between CED and EED. For the sake of clarity we shall formulate our point of view

more explicitly.

The electromagnetic field in vacuum is of discrete character and consists of single, not-interacting (or very weakly interacting) finite objects, called photons. All photons move uniformly as a whole by the same velocity c, carry finite energy E, momentum $\mathbf{p}$ *and intrinsic angular momentum. These features imply structure and internal periodic process of period T, which may be different for the various photons. The quantity $E.T$, called "elementary action", has the same value for all photons and is numerically equal to the Planck constant h. The invariance of c and h means nondistinguishability of the photons, considered as invariant objects. The integral value of the intrinsic angular momentum is equal to $\pm h$. For the topology of the 3-dimensional region, occupied by the photon at any moment, there are no experimental data, so it is desirable the model-solutions to admit arbitrary initial data.*

We'd like to stress once more: EED considers photons as *real objects*, and *not as convenient theoretical concepts*, and it aims to build adequate mathematical models of these entities. So, the first important problem is to point out the algebraic character of the modeling mathematical object for a single photon. The corresponding generalization for a number of photons is easily done (subsec.2.4).

1.5.2 Choice of the modeling mathematical object

According to the non-relativistic formulation of CED the electromagnetic field has two aspects: "electric" and "magnetic". These two aspects of the field are described by two 1-forms on $\mathcal{R}^3$ and a parametric dependence on time is admitted: the electric field E and the magnetic field B. The considerations made in (1.1.1) brought us to the conclusion that these two fields can be considered as two *vector components* of a new object, 1-form Ω, *taking values in a real 2-dimensional vector space*, naturally identified with $\mathcal{R}^2$. This mathematical object unifies and, at the same time, distinguishes the two sides of the field: there is a basis in $\mathcal{R}^2$, in which the electric and magnetic components are delimited, but in an arbitrary basis the two components mix (superimpose), so the difference between them is deleted. The physically important quantity, energy density, is given by the sum $E^2 + B^2$. This point of view seems appropriate and relevant in view of the pointed out in (1.1.1) invariance of Maxwell's equations with respect to some linear transformations, mixing E and B. All unimodular such transformations keep the energy-density unchanged.

In the relativistic formulation of CED the difference between the electric and magnetic components of the field is already quite conditional, and from invariant-theoretical point of view there is no any difference. However, the 2-component character of the field is kept in a new sense and manifests itself at a different level. In fact, as we mentioned above, some linear combinations of the electric and magnetic fields generate a new solution to Maxwell's equations. In particular, such is the transformation, defined by the matrix

$$\begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}.$$

This matrix, defining a complex structure in $\mathcal{R}^2$, transforms a field of the kind $(E,0)$ into a new field of the kind $(0,E)$ and a field of the kind $(0,B)$ into a field of the kind $(-B,0)$, i.e. the electric component into magnetic and vice versa. This observation draws our attention to looking for a complex structure $J, J^2=-id$ in the bundle of 2-forms on the Minkowski space, such that if F presents the first component of the field, then $J(F)$ to present the second component of the same field. Such complex structure truly exists and, according to (1.2.1), it coincides with the restriction of the Hodge $*$-operator, defined by the pseudometric η, to the space of 2-forms: $**_2=-id$. So, the non-relativistic vector components (E,B) are replaced by the relativistic vector components $(F,*F)$. The following considerations support also such a choice.

The relativistic Maxwell's equations in vacuum $\mathbf{d}F=0,\ \mathbf{d}*F=0$ are, obviously, invariant with respect to the interchange $F\rightarrow *F$. Moreover,if F is a solution, then an arbitrary linear combination $aF+b*F$ is again a solution. More generally, if $(F,*F)$ defines a solution, then the transformation $(F,*F)\rightarrow(aF+b*F,mF+n*F)$ defines a new solution for an arbitrary matrix

$$\begin{Vmatrix} a & m \\ b & n \end{Vmatrix}.$$

Now, using the old notation Ω for the new object $\Omega=F\otimes e_1+*F\otimes e_2$, Maxwell's equations are written down as $\mathbf{d}\Omega=0$, or equivalently $\delta\Omega=0$. Clearly, an arbitrary linear transformation of the basis (e_1,e_2) keeps Ω as a solution.

Recall the energy-momentum tensor in CED, defined by (1.7)

$$Q_\mu^\nu=\frac{1}{4\pi}\left[\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\delta_\mu^\nu-F_{\mu\sigma}F^{\nu\sigma}\right]=\frac{1}{8\pi}\left[-F_{\mu\sigma}F^{\nu\sigma}-(*F)_{\mu\sigma}(*F)^{\nu\sigma}\right].$$

It is quite clearly seen, that F and $*F$ participate in the same way in Q^μ_ν, and the full energy-momentum densities of the field are obtained through summing up the energy-momentum densities, carried by F and $*F$. Since the two expressions $F_{\mu\sigma}F^{\nu\sigma}$ and $(*F)_{\mu\sigma}(*F)^{\nu\sigma}$ are not always equal, the distribution of energy-momentum between F and $*F$ may change in time, i.e. energy-momentum may be transferred from F to $*F$, and vice versa. So we may interpret this phenomenon as a special kind of interaction between F and $*F$, responsible for some internal redistribution of the field energy. Now, in vacuum it seems naturally to expect, that the energy-momentum, carried from F to $*F$ in a given 4-volume, is the same as that, carried from $*F$ to F in the same volume. However, in presence of an active external field (medium), exchanging energy-momentum with Ω, it is hardly reasonable to trust the same expectation just because of the specific structure the external field (medium) may have. So, the external field (further any such external field will be called *medium* for short) may exchange energy-momentum preferably by F or $*F$, as well as it may support the internal redistribution of the field energy-momentum between F and $*F$, favouring F or $*F$. From the explicit form of the energy-momentum tensor it is seen that the field may participate in this exchange by means of any of the two terms $F_{\mu\sigma}F^{\nu\sigma}$ and $(*F)_{\mu\sigma}(*F)^{\nu\sigma}$. Moreover, from the expression (1.8) for the divergence of the energy-momentum tensor

$$\nabla_\nu Q^\nu_\mu = \frac{1}{4\pi}\Big[F_{\mu\nu}(\delta F)^\nu + (*F)_{\mu\nu}(\delta * F)^\nu\Big]$$

it is also clearly seen that the quantities of energy-momentum, which any of the two components F and $*F$ may exchange in a unit 4-volume are *invariantly* separated and given respectively by

$$F_{\mu\nu}(\delta F)^\nu, \quad (*F)_{\mu\nu}(\delta * F)^\nu.$$

But, in CED the exchange through $*F$ *is forbidden*, the expression $(*F)_{\mu\nu}(\delta * F)^\nu$ is always equal to zero. Of course, we do not reject the existence of such media, but we do not share the opinion that all media behave in this same way just because this can not be verified. On the other hand, in case of vacuum, we can not delimit F from $*F$, these are two solutions of the same equation and it is all the same which one will be denoted by F (or $*F$), i.e. CED does not give an intrinsic criterion for a respective choice. Only in regions with non-zero free charges and currents, when $\mathbf{d}F = 0$ and

$\delta F = j \neq 0$, the choice can be made , but this presupposes (postulates) that the field is able to interact, i.e. to exchange energy-momentum, *only* with charged particles. This postulate we can not assume ad hoc.

Having in view these considerations we assume the following postulate in EED in order to specify the algebraic character of the modeling mathematical object:

In EED the electromagnetic field is described by a 2-form Ω, defined on the Minkowski space-time and valued in a real 2-dimensional vector space $\mathcal{W}$ and such, that there is a basis (e_1, e_2) of $\mathcal{W}$ in which Ω takes the form

$$\Omega = F \otimes e_1 + *F \otimes e_2. \tag{1.37}$$

Since $\mathcal{W}$ is isomorphic to $\mathcal{R}^2$, further we shall write only $\mathcal{R}^2$ and all relations obtained can be carried over to $\mathcal{W}$ by means of the corresponding isomorphism. In particular, every $\mathcal{W}$ will be considered as being provided with a complex structure J, so, the group of automorphisms of J is defined. Our purpose now is to prove that the set of 2-forms of the kind (1.37) is stable under the invariance group of J.

First we note, that the equation $aF + b*F = 0$ requires $a = b = 0$. In fact, if $a \neq 0$ then $F = -\frac{b}{a} * F$. From $aF + b*F = 0$ we get $a*F - bF = 0$ and substituting F, we obtain $(a^2 + b^2)*F = 0$, which is possible only if $a = b = 0$ since $*F \neq 0$. In other words, F and $*F$ are linearly independent. Let now (k_1, k_2) be another basis of $\mathcal{R}^2$ and consider the 2-form $\Psi = G \otimes k_1 + *G \otimes k_2$. We express (k_1, k_2) through (e_1, e_2) and take in view what we want

$$G \otimes k_1 + *G \otimes k_2 = G \otimes (ae_1 + be_2) + *G \otimes (me_1 + ne_2) =$$

$$= (aG + m*G) \otimes e_1 + (bG + n*G) \otimes e_2 = (aG + m*G) \otimes e_1 + *(aG + m*G) \otimes e_2.$$

Consequently, $bG + n*G = a*G - mG$, i.e. $(b+m)G + (n-a)*G = 0$, which requires $m = -b,\ n = a$, i.e. the transformation matrix is

$$\left\| \begin{matrix} a & -b \\ b & a \end{matrix} \right\|.$$

Besides, if Ω_1 and Ω_2 are of the kind (1.37), it is easily shown that the linear combination $\lambda\Omega_1 + \mu\Omega_2$ is of the same kind (1.37). These results show that the 2-forms of the kind (1.37) form a stable with respect to the automorphisms of $(\mathcal{R}^2, J)$ subspace of the space $\Lambda^2(M, \mathcal{R}^2)$.

We note that the following 2-forms: $F\otimes e_1+*F\otimes e_2$, $F\otimes k_1+*F\otimes k_2$, are different, i.e. it is important which basis will be used. In order to separate the class of bases we are going to use, first we recall the product of 2 vector valued differential forms. If Φ and Ψ are respectively p and q forms on the same manifold N, taking values in the vector spaces W_1 and W_2 with corresponding bases $(e_1,...,e_m)$ and $(k_1,...,k_n)$, and $\varphi : W_1\times W_2 \to W_3$ is a bilinear map into the vector space W_3, then a $(p+q)$-form $\varphi\,(\Phi,\Psi)$ on N with values in W_3 is defined by

$$\varphi\,(\Phi,\Psi)=\sum_{i,j}\Phi^i\wedge\Psi^j\otimes\varphi(e_i,k_j).$$

In particular, if $W_1=W_2$ and $W_3=\mathcal{R}$, and the bilinear map is scalar (inner) product g, we get

$$\varphi\,(\Phi,\Psi)=\sum_{i,j}\Phi^i\wedge\Psi^j g_{ij}.$$

Let now X and Y be 2 arbitrary vector fields on the Minkowski space M, Ω be of the kind (1.37), $Q_{\mu\nu}$ be the energy tensor in CED and g be the canonical inner product in $\mathcal{R}^2$. Then the class of bases we shall use will be separated by the following equation

$$Q_{\mu\nu}X^\mu Y^\nu=\frac{1}{2}*g\Big(i(X)\Omega,*i(Y)\Omega\Big). \tag{1.38}$$

We develop the right hand side of this equation and obtain

$$\frac{1}{2}*g\Big(i(X)\Omega,*i(Y)\Omega\Big)=$$

$$\frac{1}{2}*g\Big(i(X)F\otimes e_1+i(X)*F\otimes e_2,*i(Y)F\otimes e_1+*i(Y)*F\otimes e_2\Big)=$$

$$=\frac{1}{2}*\Big[\Big(i(X)F\wedge*i(Y)F\Big)g(e_1,e_1)+\Big(i(X)F\wedge*i(Y)*F\Big)g(e_1,e_2)+$$

$$+\Big(i(X)*F\wedge*i(Y)F\Big)g(e_2,e_1)+\Big(i(X)*F\wedge*i(Y)*F\Big)g(e_2,e_2)\Big]=$$

$$=-\frac{1}{2}X^\mu Y^\nu\Big[F_{\mu\sigma}F^\sigma_\nu g(e_1,e_1)+(*F)_{\mu\sigma}(*F)^\sigma_\nu g(e_2,e_2)+$$

$$+\Big(F_{\mu\sigma}(*F)^\sigma_\nu+(*F)_{\mu\sigma}F^\sigma_\nu\Big)g(e_1,e_2)\Big]=-\frac{1}{2}X^\mu Y^\nu\Big[F_{\mu\sigma}F^\sigma_\nu+(*F)_{\mu\sigma}(*F)^\sigma_\nu\Big].$$

In order this relation to hold it is necessary to have

$$g(e_1, e_1) = 1,\ g(e_2, e_2) = 1,\ g(e_1, e_2) = 0,$$

i.e., we are going to use *orthonormal* bases. So, the stability group of the subspace of forms of the kind (1.37) is reduced to $SO(2)$ or $U(1)$. So, in this approach, the group $SO(2)$ appears in a pure algebraic way, while in the gauge interpretation of CED this group is associated with the equation $\mathbf{d}F = 0$, i.e. with the traditional and not shared bu us understanding, that the EM-field can not exchange energy-momentum with some medium through $*F$.

1.5.3 Differential equations for the field

We proceed to the main purpose, namely, to write down differential equations for our object Ω, which was chosen to model the EM-field. We shall work in the orthonormal basis (e_1, e_2), where the field has the form (1.37). The two vectors of this basis define two mutually orthogonal subspaces $\{e_1\}$ and $\{e_2\}$, such that the space $\mathcal{R}^3$ is a direct sum of these two subspaces: $\mathcal{R}^2 = \{e_1\} \oplus \{e_2\}$. So, we have the two projection operators $\pi_1 : \mathcal{R}^2 \rightarrow \{e_1\},\ \pi_2 : \mathcal{R}^2 \rightarrow \{e_2\}$. These two projection operators extend to projections in the $\mathcal{R}^2$-valued differential forms on M:

$$\pi_1\Omega = \pi_1(\Omega^1 \otimes k_1 + \Omega^2 \otimes k_2) = \Omega^1 \otimes \pi_1 k_1 + \Omega^2 \otimes \pi_1 k_2 =$$

$$= \Omega^1 \otimes \pi_1(ae_1 + be_2) + \Omega^2 \otimes \pi_1(me_1 + ne_2) = (a\Omega^1 + m\Omega^2) \otimes e_1.$$

Similarly,

$$\pi_2\Omega = (b\Omega_1 + n\Omega_2) \otimes e_2.$$

In particular, if Ω is of the form (1.37), then

$$\pi_1(F \otimes e_1 + *F \otimes e_2) = F \otimes e_1,\ \pi_2(F \otimes e_1 + *F \otimes e_2) = *F \otimes e_2.$$

Let now our EM-field Ω propagates in a region, where some other continuous physical object (external field, medium) also propagates and exchanges energy-momentum with Ω. We are going to define explicitly the local law this exchange obeys.

First we note, that the external field is described by some mathematical object(s). From this mathematical object, following definite rules, reflecting

the specific situation under consideration, a new mathematical object $\mathcal{A}_i$ is constructed and this new mathematical object participates directly in the exchange defining expression. The EM-field participates in this exchange defining expression directly through Ω, and since Ω takes values in $\mathcal{R}^2$, then $\mathcal{A}_i$ must also take values in $\mathcal{R}^2$.

We make now two preliminary remarks. First, all operators, acting on the usual differential forms, are naturally extended to act on vector valued differential forms according to the rule $D \to D \times id$. In particular,

$$*\Omega = *(\sum_i \Omega^i \otimes e_i) = \sum_i (*\Omega^i) \otimes e_i, \ \mathbf{d}\Omega = \mathbf{d}(\sum_i \Omega^i \otimes e_i) = \sum_i (\mathbf{d}\Omega^i) \otimes e_i,$$

$$\delta\Omega = \delta(\sum_i \Omega^i \otimes e_i) = \sum_i (\delta\Omega^i) \otimes e_i.$$

Second, in view of the importance of the expression (1.8) for the divergence of the CED energy-momentum tensor, we give its explicit deduction. Recall the following algebraic relations on the Minkowski space:

$$\delta_p = (-1)^p *^{-1} \mathbf{d}* = *\mathbf{d}*, \ \delta*_p = *\mathbf{d}_p \ for \ p = 2k+1, \ \delta*_p = -*\mathbf{d}_p \ for \ p = 2k. \tag{1.39}$$

If α is a 1-form and F is a 2-form, the following relations hold:

$$*\,(\alpha \wedge *F) = -\alpha^\mu F_{\mu\nu} dx^\nu = *\,[(*F) \wedge *(*\alpha)] = \frac{1}{2}(*F)^{\mu\nu}(*\alpha)_{\mu\nu\sigma} dx^\sigma. \tag{1.40}$$

In particular,

$$*(F \wedge *\mathbf{d}F) = \frac{1}{2} F^{\mu\nu}(\mathbf{d}F)_{\mu\nu\sigma} dx^\sigma = *[\delta * F \wedge *(*F)] = (*F)_{\mu\nu}(\delta * F)^\nu dx^\mu.$$

Having in view these relations, we obtain consecutively:

$$\nabla_\nu Q^\nu_\mu = \nabla_\nu \left[\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \delta^\nu_\mu - F_{\mu\sigma} F^{\nu\sigma}\right] =$$

$$= \frac{1}{2} F^{\alpha\beta} \nabla_\nu F_{\alpha\beta} \delta^\nu_\mu - (\nabla_\nu F_{\mu\sigma}) F^{\nu\sigma} - F_{\mu\sigma} \nabla_\nu F^{\nu\sigma} =$$

$$= \frac{1}{2} F^{\alpha\beta} [(\mathbf{d}F)_{\alpha\beta\mu} - \nabla_\alpha F_{\beta\mu} - \nabla_\beta F_{\mu\alpha}] - (\nabla_\nu F_{\mu\sigma}) F^{\nu\sigma} - F_{\mu\sigma} \nabla_\nu F^{\nu\sigma} =$$

$$= \frac{1}{2} F^{\alpha\beta} (\mathbf{d}F)_{\alpha\beta\mu} - F_{\mu\sigma} \nabla_\nu F^{\nu\sigma} = -(*F)_{\mu\nu} \nabla_\sigma (*F)^{\sigma\nu} - F_{\mu\nu} \nabla_\sigma F^{\sigma\nu} =$$

$$= (*F)_{\mu\nu}(\delta * F)^\nu + F_{\mu\nu}(\delta F)^\nu.$$

Let now our field Ω interact with some other field. This interaction, i.e. energy-momentum exchange, is performed along 3 "channels". The first 2 channels are defined by the 2 (equal in rights) components F and $*F$ of Ω. This exchange is *real* in the sense, that some part of the EM-energy-momentum may be transformed into some other kind of energy-momentum and assimilated by the external field or dissipated. Since the two components F and $*F$ are equal in rights it is naturally to expect that the corresponding 2 terms, defining the exchange in a unit 4-volume, will depend on F and $*F$ similarly. The above expression for $\nabla_\nu Q_\mu^\nu$ gives the two 1-forms

$$F_{\mu\nu}(\delta F)^\nu dx^\mu, \quad (*F)_{\mu\nu}(\delta * F)^\nu dx^\mu$$

as natural candidates for this purpose. As for the third channel, it takes into account a possible influence of the external field on the intra-field exchange between F and $*F$, which occurs without assimilation of energy-momentum by the external field. The natural candidate, describing this exchange is, obviously, the expression

$$F_{\mu\nu}(\delta * F)^\nu dx^\mu + (*F)_{\mu\nu}(\delta F)^\nu dx^\mu.$$

It is important to note, that these three channels are independent in the sense, that any of them may occur without taking care if the other two work or don't work. A natural model for such a situation is a 3-dimensional vector space K, where the three dimensions correspond to the three exchange channels. The non-linear exchange law requires some K-valued non-linear map. Since our fields take values in $\mathcal{R}^2$ this 3-dimensional space must be constructed from $\mathcal{R}^2$ in a natural way. Having in view the bilinear character of $\nabla_\nu Q_\mu^\nu$ it seems naturally to look for some bilinear construction with the properties desired. These remarks suggest to choose the *symmetrized tensor product* $Sym(\mathcal{R}^2 \otimes \mathcal{R}^2) \equiv \mathcal{R}^2 \vee \mathcal{R}^2$. So, from the point of view of the EM-field, the energy-momentum exchange term should be written in the following way:

$$\vee\, (\delta\Omega, *\Omega). \tag{1.41}$$

In fact, in the corresponding basis (e_1, e_2) we obtain

$$\vee(\delta\Omega, *\Omega) = \vee(\delta F \otimes e_1 + \delta * F \otimes e_2, *F \otimes e_1 + ** F \otimes e_2) =$$

$$= (\delta F \wedge *F) \otimes e_1 \vee e_1 + (\delta * F \wedge ** F) \otimes e_2 \vee e_2 + (-\delta F \wedge F + \delta * F \wedge *F) \otimes e_1 \vee e_2.$$

This expression determines how much of the EM-field energy-momentum has been carried irreversibly over to the external field and how much has been redistributed between F and $*F$ by virtue of the external field influence in a unit 4-volume.

Now, this same quantity of energy-momentum has to be expressed by new terms, in which the external field "agents" should participate. Let's denote by Φ the first agent, interacting with $\pi_1\Omega$, and by Ψ the second agent, interacting with $\pi_2\Omega$. Since the corresponding two exchanges are independent, we may write the exchange term in the following way:

$$\vee\,(\Phi, *\pi_1\Omega) \ + \ \vee(\Psi, *\pi_2\Omega). \tag{1.42}$$

According to the local energy-momentum conservation law these two quantities have to be equal, so we obtain

$$\vee\,(\delta\Omega, *\Omega) = \vee(\Phi, *\pi_1\Omega) \ + \ \vee(\Psi, *\pi_2\Omega). \tag{1.43}$$

This is the basic relation in EED. It contains the basic differential equations for the EM-field components and requires additional equations, specifying the properties of the external field, i.e. the algebraic and differential properties of Φ and Ψ. The physical sense of this equation is quite clear: local balance of energy-momentum. Further we shall study this relation in various cases, and in the first place - the vacuum.

Chapter 2

Extended Electrodynamics in Vacuum

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2.1 Vacuum Equations in EED

2.1.1 Vacuum in EED

In CED the term "vacuum" is used in the sense, that in the region, where we consider an EM-field, there is no free or bound electric charges. This notion of vacuum in CED comes from the conception, that the field may exchange energy-momentum *only* with electric charge carriers and *only* through the component F. EED extends the possibilities for energy-momentum exchange assuming that the full quantity of exchanged energy-momentum in a unit 4-volume is given by the general expression (1.42). That's why in EED we talk about a field Ω in vacuum every time when this expression (1.42) is equal to zero. Formally this means that every external field (medium), which does not exchange energy-momentum with Ω can be treated as vacuum as far as the EM-field is concerned. Talking about exchange, we mostly have in mind that some energy-momentum is transferred from the field to the medium, however, we do not formally exclude the reverse process.

Assume now that in some region we have an EM-field Ω. In the corresponding basis (e_1, e_2) we can write

$$\Omega = F \otimes e_1 + *F \otimes e_2,\ \Phi = \alpha^1 \otimes e_1 + \alpha^2 \otimes e_2,\ \Psi = \alpha^3 \otimes e_1 + \alpha^4 \otimes e_2.$$

Remark. Further the 1-forms $\alpha^i,\ i = 1, ...4$, as well as the corresponding through the pseudometric η vector fields, will be called shortly *currents.*

Of course, this terminology should not be associated with some charged particles, or with some before given specific structure. In the general case these currents are just the tools of the external field to gain some energy-momentum from the field Ω.

The expression (1.42) looks as follows:

$$\vee(\Phi, *\pi_1\Omega) + \vee(\Psi, *\pi_2\Omega) = \alpha^1 \wedge *F \otimes e_1 \vee e_1 + \alpha^4 \wedge * * F \otimes e_2 \vee e_2 +$$

$$+(\alpha^3 \wedge * * F + \alpha^2 \wedge *F) \otimes e_1 \vee e_2.$$

Clearly. the necessary and sufficient condition for vacuum is

$$\alpha^1 \wedge *F = 0,\ \alpha^4 \wedge * * F = 0,\ \alpha^3 \wedge * * F + \alpha^2 \wedge *F = 0, \tag{2.1}$$

or, in components

$$F_{\mu\nu}\alpha_1^\nu = 0,\ (*F)_{\mu\nu}\alpha_4^\nu = 0,\ (*F)_{\mu\nu}\alpha_3^\nu + F_{\mu\nu}\alpha_2^\nu = 0. \tag{2.2}$$

We see that there are various possibilities, i.e. relations among F and α_i, to realize a vacuum situation. The strongest condition is, of course, $\Phi = \Psi = 0$, i.e. all currents are equal to zero: $\alpha^i = 0,\ i = 1, ..., 4$. If, at least one of the two currents α^1 and α^2 is different from zero, then the above relations (1.45) require $det\|F_{\mu\nu}\| = 0$, i.e. $F \wedge F = 0$, or orthogonality between E and B. If $\alpha^4 = \alpha^1 = 0$, but the other two currents are different from zero then $F \wedge F \neq 0$ in general. In accordance with our interpretation of the equations (1.42)this means that the medium affects the exchange between F and $*F$ without gaining and keeping any energy-momentum. In such a case the orthogonality between E and B is not needed. It seems important to note, that (2.1) requires every couple α^i, α^j to be in the kernel of F and $*F$, i.e.

$$F(\alpha^i, \alpha^j) = (*F)(\alpha^i, \alpha^j) = 0. \tag{2.3}$$

So, in case of vacuum, the currents are strongly dependent on the field Ω. The above relations define equations, connecting the 16 components of the 4 currents with the components $F_{\mu\nu}$. Of course, in curvelinear coordinates $\{y^\sigma\}$ these equations will depend strongly on the metric coefficients $\eta_{\mu\nu}(y^\sigma)$ in these coordinates. Finally we note, that α^1 and α^4 may be considered as eigen vectors respectively for F and $*F$ at zero eigen values, which, according to the formulas in (1.1.2), is always possible if $I_2 = 2E.B = 0$.

2.1.2 Equivalent forms of the equations

According to the last subsection an external field is called vacuum with respect to the EM-field Ω if the right hand side of (1.43) is equal to zero. Then the left hand side of (1.43) will also be equal to zero, so we get the equations

$$\vee(\delta\Omega, *\Omega) = 0. \tag{2.4}$$

From this coordinate free compactly written expression we obtain the following equations for the components of Ω in the basis (e_1, e_2):

$$\begin{gathered}(\delta F \wedge *F)\otimes e_1 \vee e_1 + (\delta * F) \wedge * * F) \otimes e_2 \vee e_2 + \\ +(\delta F \wedge * * F + \delta * F \wedge *F) \otimes e_1 \vee e_2 = 0.\end{gathered} \tag{2.5}$$

So, the field equations, expressed through the operator δ, look as follows

$$\delta F \wedge *F = 0,\ \delta * F \wedge * * F = 0,\ \delta * F \wedge *F - \delta F \wedge F = 0. \tag{2.6}$$

These equations, expressed through the operator $\mathbf{d}$, have the form

$$* F \wedge *\mathbf{d} * F = 0,\ F \wedge *\mathbf{d}F = 0,\ F \wedge *\mathbf{d} * F + *F \wedge *\mathbf{d}F = 0. \tag{2.7}$$

Using the components $F_{\mu\nu}$, we obtain for the equations (2.6)

$$F_{\mu\nu}(\delta F)^\nu = 0,\ (*F)_{\mu\nu}(\delta * F)^\nu = 0,\ F_{\mu\nu}(\delta * F)^\nu + (*F)_{\mu\nu}(\delta F)^\nu = 0. \tag{2.8}$$

In the same way, for the equations (2.7) we get

$$(*F)^{\mu\nu}(\mathbf{d} * F)_{\mu\nu\sigma} = 0,\ F^{\mu\nu}(\mathbf{d}F)_{\mu\nu\sigma} = 0,\ (*F)^{\mu\nu}(\mathbf{d}F)_{\mu\nu\sigma} + F^{\mu\nu}(\mathbf{d} * F)_{\mu\nu\sigma} = 0. \tag{2.9}$$

Now we give the 3-dimensional form of the equations in the same order:

$$B \times \left(rotB - \frac{\partial E}{\partial \xi}\right) - EdivE = 0,\ E.\left(rotB - \frac{\partial E}{\partial \xi}\right) = 0, \tag{2.10}$$

$$E \times \left(rotE + \frac{\partial B}{\partial \xi}\right) - BdivB = 0,\ B.\left(rotE + \frac{\partial B}{\partial \xi}\right) = 0, \tag{2.11}$$

$$\left(rotE + \frac{\partial B}{\partial \xi}\right) \times B + \left(rotB - \frac{\partial E}{\partial \xi}\right) \times E + BdivE + EdivB = 0,$$

$$B.\left(rotB-\frac{\partial E}{\partial \xi}\right)-E.\left(rotE+\frac{\partial B}{\partial \xi}\right)=0. \qquad (2.12)$$

From the second equations of (2.10) and (2.11) the well known Poynting relation follows

$$div\,(E\times B)+\frac{\partial}{\partial \xi}\frac{E^2+B^2}{2}=0,$$

and from the second equation of (2.12), if $E.B=g(x,y,z)$, we obtain the known from Maxwell theory relation

$$B.rotB=E.rotE.$$

The explicit form of equations (2.10) end (2.11) should not make us conclude, that the second (scalar) equations follow from the first (vector) equations. Here is an example: let $divE=0,\ divB=0$ and the time derivatives of E and B are zero. Then the system of equations reduces to

$$E\times rotE=0,\ B\times rotB=0,\ B.rotE=0,\ E.rotB=0.$$

As it is seen, the vector equations do not require any connection between E and B in this case, therefore, the scalar equations, which impose such a connection, can not follow from the vector ones. The third equations of (2.7) and (2.8) determine (in equivalent way) the energy-momentum quantities, transferred from F to $*F$, and reversely, in a unit 4-volume, with the expressions, respectively

$$F_{\mu\nu}(\delta*F)^\nu=-(*F)_{\mu\nu}(\delta F)^\nu,\ F^{\mu\nu}(\mathbf{d}*F)_{\mu\nu\sigma}=-(*F)^{\mu\nu}(\mathbf{d}F)_{\mu\nu\sigma}.$$

From these relations it is seen, that the 1-forms δF and $\delta*F$ *play the role of "external" currents* respectively for $*F$ and F. In the same spirit we could say, that the energy-momentum quantities $F_{\mu\nu}(\delta F)^\nu$ and $(*F)_{\mu\nu}(\delta*F)^\nu$, which F and $*F$ exchange with themselves, are equal to zero. And this corresponds fully to our former statements, concerning the physical sense of the equations for the components of Ω.

2.1.3 Conservation laws

From the first two equations of (2.8) and from the earlier given expression for the divergence $\nabla_\nu Q_\mu^\nu$ of the Maxwell's energy-momentum tensor Q_μ^ν in CED

it is immediately seen that on the solutions of our equations (2.8) this divergence is also zero. In view of this *we assume the tensor* Q_μ^ν, *defined by* (1.7) *to be the energy-momentum tensor in* EED. We shall be interested in finding explicit time-stable solutions of finite type, i.e. $F_{\mu\nu}$ to be finite functions of the three spatial coordinates, therefore, if it turns out that such solutions really exist, then integral conserved quantities can be easily constructed and computed, making use of the 10 Killing vectors on the Minkowski space-time. We recall that in CED such finite and time-stable solutions in the whole space are not allowed by the Maxwell's equations.

2.2 General Properties of the Equations and Their Solutions

2.2.1 General properties of the equations

We first note, that in correspondence with the requirement for *general covariance*, equations (2.4), given above and presented in different but equivalent forms, are written down in coordinate free manner. This requirement is universal, i.e. it concerns all basic equations of a theory and means simply, that the existence of real objects and the occurrence of real processes *can not* depend on the local coordinates used in the theory., i.e. on the convenient for us way to describe the local character of the evolution and structure of the natural objects and processes. Of course, in the various coordinate systems the equations and their solutions will look differently. Namely the covariant character of the equations allows to choose the most appropriate coordinates, reflecting most fully the features of every particular case. A typical example for this is the usage of spherical coordinates in describing spherically symmetric fields. Let's not forget also, that the coordinate-free form of the equations permits an easy transfer of the same physical situation onto manifolds with more complicated structure and nonconstant metric tensor. Shortly speaking, *the coordinate free form of the equations in theoretical physics reflects the most essential properties of reality, called shortly objective character of the real phenomena.*

Since the left hand sides of the equations are linear combinations of the first derivatives of the unknown functions with coefficients, depending linearly on these unknown functions, (2.8) present a special type system of *quasilinear first order partial differential equations.* The number of the unknown

functions $F_{\mu\nu} = -F_{\nu\mu}$ is 6, and in general, the number of the equations is $3.4 = 12$, but the number of the independent equations depends strongly on if the two invariants $I_1 = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}$ and $I_2 = \frac{1}{2}(*F)_{\mu\nu}F^{\mu\nu}$ are equal to zero or not equal to zero. If $I_2 \neq 0$ then $det(F_{\mu\nu}) \neq 0$ and the first two equations of (2.8) are equivalent to $\delta F = 0$ and $\delta * F = 0$, which automatically eliminates the third equation of (2.8), i.e. in this case our equations reduce to Maxwell's equations.

It is clearly seen from the (2.9) form of the equations, that the metric tensor essentially participates (through the $*$-operator applied to 2-forms only) in the equations. If we use the δ-operator, then the metric participates also through the $*$-operator, applied to 3-forms, but this does not lead to more complicated coordinate form of the equations. It is worth to note that in nonlinear coordinates the metric tensor will participate with its derivatives, therefore, the very solutions will depend strongly on the metric tensor chosen. This may cause existence or non-existence of solutions of a given class, e.g. soliton-like ones. In our framework such additional complications do not appear because of the opportunity to work in global coordinates with constant metric tensor.

We note 2 important invariance (symmetry) properties of our equations.

Property 1. *The transformation $F \rightarrow *F$ does not change the system.*

The proof is obvious, in fact, the first two equations interchange, and the third one is kept the same. In terms of Ω this means that if Ω is a solution, then $*\Omega$ is also a solution, which means, in turn, that equations (2.4) are equivalent to equations

$$\vee(\Omega, *\mathbf{d}\Omega) = 0. \tag{2.13}$$

Property 2. *Under conformal change of the metric the equations do not change.*

The proof of this property is also obvious and is reduced to the notice, that as it is seen from (2.9), the $*$-operator participates only with its reduction to 2-forms, and as we noted in subsec.(1.1.2), $*_2$ is conformally invariant.

Summing up the first two equations of (2.10) and (2.11) we obtain how the classical Poynting vector changes in time in our more general approach:

$$\frac{\partial}{\partial \xi}(E \times B) = EdivE + BdivB - E \times rotE - B \times rotB.$$

In CED the first and the second terms on the right are missing.

Here is an example of static solutions of (2.4), which are not solutions to Maxwell's equations.

$$E = (a sin\alpha z,\ a cos\alpha z,\ 0),\ B = (b cos\alpha z,\ -b sin\alpha z,\ 0),$$

where a, b and α are constants. We obtain

$$rotE = (a\alpha sin\alpha z,\ a\alpha cos\alpha z,\ 0),\ rotB = (b\alpha cos\alpha z,\ -b\alpha sin\alpha z,\ 0)$$

Obviously, $E \times rotE = 0,\ B \times rotB = 0,\ E.rotB = 0,\ B.rotE = 0$. For the Poynting vector we get $E \times B = (0, 0, -ab)$, and for the energy density $w = \frac{1}{2}(a^2 + b^2)$. Considered in a finite volume, these solutions could model some standing waves, but we do not engage ourselves with such interpretations, since we do not accept seriously that static EM-fields may really exist.

2.2.2 General properties of the solutions

It is quite clear that the solutions of our equations are naturally divided into two classes: *linear* and *nonlinear*. The first class consists of all solutions to Maxwell's vacuum equations, where the name *linear* comes from. These solutions are well known and won't be discussed here. The second class, called *nonlinear*, includes all the rest solutions. This second class is naturally divided into two subclasses. The first subclass consists of all nonlinear solutions, satisfying the conditions

$$\delta F \neq 0,\ \delta * F \neq 0, \tag{2.14}$$

and the second subclass consists of those nonlinear solutions, satisfying one of the two couples of conditions:

$$\delta F = 0,\ \delta * F \neq 0;\ \delta F \neq 0,\ \delta * F = 0.$$

Further we assume the conditions (2.14) fulfilled, i.e. the solutions of the second subclass will be considered as particular cases of the first subclass. Our purpose is to show explicitly, that among the nonlinear solutions there are soliton-like ones, i.e. the components $F_{\mu\nu}$ of which at any moment are *finite* functions of the three spatial variables with *connected* support. We are going to study their properties and to introduce corresponding characteristics. First we shall establish some of their basic features, proving three propositions.

Proposition 1. *All nonlinear solutions have zero invariants*:

$$I_1=\frac{1}{2}F_{\mu\nu}F^{\mu\nu}=0,\ I_2=\frac{1}{2}(*F)_{\mu\nu}F^{\mu\nu}=2\sqrt{det(F_{\mu\nu})}=0.$$

Proof. Recall the field equations in the form (2.8):

$$F_{\mu\nu}(\delta F)^\nu=0,\ (*F)_{\mu\nu}(\delta*F)^\nu=0,\ F_{\mu\nu}(\delta*F)^\nu+(*F)_{\mu\nu}(\delta F)^\nu=0.$$

It is clearly seen that the first two groups of these equations may be considered as two linear homogeneous systems with respect to δF^μ and $\delta*F^\mu$ respectively. In view of the nonequalities (2.14) these homogeneous systems have non-zero solutions, which is possible only if $det(F_{\mu\nu})=det(*F)_{\mu\nu})=0$, i.e. if $I_2=2E.B=0$. Further, summing up these three systems of equations, we obtain

$$(F+*F)_{\mu\nu}(\delta F+\delta*F)^\nu=0.$$

If now $(\delta F+\delta*F)^\nu\neq 0$, then

$$0=det(F+*F)_{\mu\nu}=\left[\frac{1}{2}(F+*F)_{\mu\nu}(*F-F)^{\mu\nu}\right]^2=\frac{1}{4}\left[-2F_{\mu\nu}F^{\mu\nu}\right]^2=(I_1)^2.$$

If $\delta F^\nu=-(\delta*F)^\nu\neq 0$, we sum up the first two systems and obtain $(*F-F)_{\mu\nu}(\delta*F)^\nu=0$. Consequently,

$$0=det(*F-F)_{\mu\nu}=\left[\frac{1}{2}(*F-F)_{\mu\nu}(-F-*F)^{\mu\nu}\right]^2=\frac{1}{4}\left[2F_{\mu\nu}F^{\mu\nu}\right]^2=(I_1)^2.$$

This completes the proof.

Recall that in this case the energy-momentum tensor $Q_{\mu\nu}$ has just one isotropic eigen direction and all other eigen directions are space-like. Since all eigen directions of $F_{\mu\nu}$ and $*F_{\mu\nu}$ are eigen directions of $Q_{\mu\nu}$ too, it is clear that $F_{\mu\nu}$ and $(*F)_{\mu\nu}$ can not have time-like eigen directions. But the first two systems of (2.8) require δF and $\delta*F$ to be eigen vectors of F and $*F$ respectively, so we obtain

$$(\delta F).(\delta F)\leq 0,\ (\delta*F).(\delta*F)\leq 0. \tag{2.15}$$

Proposition 2. *All nonlinear solutions satisfy the conditions*

$$(\delta F)_\mu(\delta*F)^\mu=0,\ |\delta F|=|\delta*F| \tag{2.16}$$

Proof. We form the inner product $i(\delta * F)(\delta F \wedge *F) = 0$ and get

$$(\delta * F)^\mu (\delta F)_\mu (*F) - \delta F \wedge (\delta * F)^\mu (*F)_{\mu\nu} dx^\nu = 0.$$

Because of the obvious nulification of the second term the first term will be equal to zero (at non-zero $*F$) only if $(\delta F)_\mu (\delta * F)^\mu = 0$.

Further we form the inner product $i(\delta * F)(\delta F \wedge F - \delta * F \wedge *F) = 0$ and obtain

$$(\delta * F)^\mu (\delta F)_\mu F - \delta F \wedge (\delta * F)^\mu F_{\mu\nu} dx^\nu -$$
$$-(\delta * F)^2 (*F) + \delta * F \wedge (\delta * F)^\mu (*F)_{\mu\nu} dx^\nu = 0.$$

Clearly, the first and the last terms are equal to zero. So, the inner product by δF gives

$$(\delta F)^2 (\delta * F)^\mu F_{\mu\nu} dx^\nu - [(\delta F)^\mu (\delta * F)^\nu F_{\mu\nu}] \, \delta F + (\delta * F)^2 (\delta F)^\mu (*F)_{\mu\nu} dx^\nu = 0.$$

The second term of this equality is zero. Besides, $(\delta * F)^\mu F_{\mu\nu} dx^\nu = -(\delta F)^\mu (*F)_{\mu\nu} dx^\nu$. So,

$$\left[(\delta F)^2 - (\delta * F)^2\right] (\delta F)^\mu (*F)_{\mu\nu} dx^\nu = 0.$$

Now, if $(\delta F)^\mu (*F)_{\mu\nu} dx^\nu \neq 0$, then the relation $|\delta F| = |\delta * F|$ follows immediately. If $(\delta F)^\mu (*F)_{\mu\nu} dx^\nu = 0 = -(\delta * F)^\mu F_{\mu\nu} dx^\nu$ according to the third equation of (2.8), we shall show that $(\delta F)^2 = (\delta * F)^2 = 0$. In fact, forming the inner product $i(\delta F)(\delta F \wedge *F) = 0$, we get

$$(\delta F)^2 * F - \delta F \wedge (\delta F)^\mu (*F)_{\mu\nu} dx^\nu = (\delta F)^2 * F = 0.$$

In a similar way, forming the inner product $i(\delta * F)\delta * F \wedge F = 0$ we have

$$(\delta * F)^2 F - \delta(*F) \wedge (\delta * F)^\mu F_{\mu\nu} dx^\nu = (\delta * F)^2 F = 0.$$

This completes the proof.
We just note that in this last case the isotropic vectors δF and $\delta * F$ are eigen vectors of $Q_{\mu\nu}$ too, and since $Q_{\mu\nu}$ has just one isotropic eigen direction, we conclude that δF and $\delta * F$ are colinear.

In order to formulate the third proposition, we recall from subsec. (1.1.2) that at zero invariants $I_1 = I_2 = 0$ the following representation holds:

$$F = A \wedge \zeta, \; *F = A^* \wedge \zeta,$$

where ζ is the only (up to a scalar multiple) isotropic eigen vector of Q_μ^ν. Also, the relations $A.\zeta = 0, \;\; A^*.\zeta = 0$ are in force. Having this in view we shall prove the following

Proposition 3. *All nonlinear solutions satisfy the relations*

$$\zeta^\mu(\delta F)_\mu = 0,\ \zeta^\mu(\delta * F)_\mu = 0. \tag{2.17}$$

Proof. We form the inner product $i(\zeta)(\delta F \wedge *F) = 0$:

$$[\zeta^\mu(\delta F)_\mu] * F - \delta F \wedge (\zeta)^\mu(*F)_{\mu\nu}dx^\nu =$$

$$= [\zeta^\mu(\delta F)_\mu]\, A^* \wedge \zeta - (\delta F \wedge \zeta)\zeta^\mu(A^*)_\mu + (\delta F \wedge A^*)\zeta^\mu\zeta_\mu = 0.$$

Since the second and the third terms are equal to zero and $*F \neq 0$, then $\zeta^\mu(\delta F)_\mu = 0$. Similarly, from the equation $(\delta * F) \wedge F = 0$ we get $\zeta^\mu(\delta * F)_\mu = 0$. The proposition is proved.

2.2.3 Algebraic properties of the nonlinear solutions

Since all nonlinear solutions have zero invariants $I_1 = I_2 = 0$ we can make a number of algebraic considerations, which clarify considerably the structure and make easier the study of the properties of these solutions. As we mentioned earlier, all eigen values if F, $*F$ and $Q_{\mu\nu}$ in this case are zero, and the eigen vectors can not be time-like. There is only one isotropic direction, defined by the isotropic vectors $\pm\zeta$ and the representations $F = A \wedge \zeta$, $*F = A^* \wedge \zeta$ hold, moreover, we have $A.A^* = 0,\ A^2 = (A^*)^2 \leq 0, A.\zeta = A^*.\zeta = 0$. Recall that the two 1-forms A and A^* are defined up to isotropic additive factors, colinear to ζ. The above representation of F and $*F$ through ζ shows that these factors do not contribute to F and $*F$, therefore, we assume further that, *these additive factors are equal to zero.*

We express now $Q_{\mu\nu}$ through A, A^* and ζ. First we normalize the vector ζ. This is possible, because it is an isotropic vector, so its time-like component ζ_4 is always different from zero. We divide ζ_μ by ζ_4 and get the vector $\mathbf{V} = (\mathbf{V}^1, \mathbf{V}^2, \mathbf{V}^3, 1)$, defining, of course, the same isotropic direction. Now we make use of the identity (1.25), where we put $F_{\mu\nu}$ instead of $G_{\mu\nu}$. Having in view that $I_1 = \frac{1}{2}F_{\mu\nu}F^{\mu\nu} = 0$, we obtain $F_{\mu\sigma}F^{\nu\sigma} = (*F)_{\mu\sigma}(*F)^{\nu\sigma}$. So, the energy-momentum tensor looks as follows

$$Q_\mu^\nu = -\frac{1}{4\pi}F_{\mu\sigma}F^{\nu\sigma} = -\frac{1}{4\pi}(*F)_{\mu\sigma}(*F)^{\nu\sigma} =$$

$$= -\frac{1}{4\pi}(A)^2\mathbf{V}_\mu\mathbf{V}^\nu = -\frac{1}{4\pi}(A^*)^2\mathbf{V}_\mu\mathbf{V}^\nu. \tag{2.18}$$

This choice of $\zeta = \mathbf{V}$ determines the following energy density $4\pi Q_4^4 = |A|^2 = |A^*|^2$.

We consider now the influence of the conservation law $\nabla_\nu Q_\mu^\nu = 0$ on $\mathbf{V}$.

$$\nabla_\nu Q_\mu^\nu = -A^2 \mathbf{V}^\nu \nabla_\nu \mathbf{V}_\mu - \mathbf{V}_\mu \nabla_\nu \left(A^2 \mathbf{V}^\nu\right) = 0.$$

This relation holds for every $\mu = 1,2,3,4$. We consider it for $\mu = 4$ and get $\mathbf{V}^\nu \nabla_\nu(1) = \mathbf{V}^\nu \partial_\nu(1) = 0$. Therefore, $\mathbf{V}_4 \nabla_\nu \left(A^2 \mathbf{V}^\nu\right) = \nabla_\nu \left(A^2 \mathbf{V}^\nu\right) = 0$. Since $A^2 \neq 0$, we obtain that $\mathbf{V}$ satisfies the equation

$$\mathbf{V}^\nu \nabla_\nu \mathbf{V}^\mu = 0,$$

which means, that $\mathbf{V}$ is a *geodesic* vector field, i.e. the integral trajectories of $\mathbf{V}$ are isotropic geodesics, or isotropic straight lines. Hence, *every nonlinear solution F defines unique isotropic geodesic direction in the Minkowski space-time*. This important consequence allows a special class of coordinate systems, called further F-adapted, to be introduced. These coordinate systems are defined by the requirement, that the trajectories of the unique $\mathbf{V}$, defined by F, to be parallel to the (z,ξ)-coordinate plane. In such a coordinate system we have $\mathbf{V}_\mu = (0,0,\varepsilon,1),\ \varepsilon = \pm 1$. Further on, we shall work in such arbitrary chosen but fixed F-adapted coordinate system, defined by the corresponding F under consideration.

We write down now the relations $F = A \wedge \mathbf{V},\ *F = A^* \wedge \mathbf{V}$ componentwise, take into account the values of $\mathbf{V}_\mu$ in the F-adapted coordinate system and obtain the following explicit relations:

$$F_{12} = F_{34} = 0,\ F_{13} = \varepsilon F_{14},\ F_{23} = \varepsilon F_{24},$$

$$(*F)_{12} = (*F)_{34} = 0,\ (*F)_{13} = \varepsilon(*F)_{14} = -F_{24},\ (*F)_{23} = \varepsilon(*F)_{24} = F_{14},$$

$$A = \left(F_{14}, F_{24}, 0, 0\right),\ A^* = \left(-F_{23}, F_{13}, 0, 0\right) = \left(-\varepsilon A_2, \varepsilon A_1, 0, 0\right). \qquad (2.19)$$

Clearly, the 1-forms A and $-A^*$ can be interpreted as *electric* and *magnetic* fields respectively. Only 4 of the components Q_μ^ν are different from zero, namely: $Q_4^4 = -Q_3^3 = \varepsilon Q_3^4 = -\varepsilon Q_4^3 = |A^2|$. Introducing the notations $F_{14} \equiv u,\ F_{24} \equiv p$, we can write

$$F = \varepsilon u dx \wedge dz + u dx \wedge d\xi + \varepsilon p dy \wedge dz + p dy \wedge d\xi$$

$$*F = -p dx \wedge dz - \varepsilon p dx \wedge d\xi + u dy \wedge dz + \varepsilon u dy \wedge d\xi.$$

In the important for us *spatially finite* case, i.e. when the functions u and p are finite with respect to the spatial variables (x, y, z), for the integral energy W and momentum $\mathbf{p}$ we obtain

$$W = \int Q_4^4 dxdydz = \int (u^2 + p^2) dxdydz < \infty,$$

$$\mathbf{p} = \left(0, 0, \varepsilon \frac{W}{c}\right), \ \rightarrow c^2|\mathbf{p}|^2 - W^2 = 0. \qquad (2.20)$$

Now we show how the nonlinear solution F defines at every point a pseudoorthonormal basis in the corresponding tangent and cotangent spaces. The nonzero 1-forms A and A^* are normed to $\mathbf{A} = A/|A|$ and $\mathbf{A}^* = A^*/|A^*|$. Two new unit 1-forms $\mathbf{R}$ and $\mathbf{S}$ are introduced through the equations:

$$\mathbf{R}^2 = -1, \ \mathbf{A}^\nu \mathbf{R}_\nu = 0, \ (\mathbf{A}^*)^\nu \mathbf{R}_\nu = 0, \ \mathbf{V}^\nu \mathbf{R}_\nu = \varepsilon, \ \mathbf{S} = \mathbf{V} + \varepsilon \mathbf{R}.$$

The only solution of the first 4 equations is $\mathbf{R}_\mu = (0, 0, -1, 0)$. Then for $\mathbf{S}$ we obtain $\mathbf{S}_\mu = (0, 0, 0, 1)$. Clearly, $\mathbf{R}^2 = -1$ and $\mathbf{S}^2 = 1$. This pseudoorthonormal (co-tangent) basis is carried over to a (tangent) pseudoorthonormal basis by means of the pseodometric η.

We proceed further to introduce the concepts of *amplitude* and *phase* in a coordinate-free manner. Of course, we shall use the considerations in subsec.(1.3.1). First, of course, we look at the invariants, we have: $I_1 = I_2 = 0$. But in our case we have got another invariant, namely, the module of the 1-forms A and A^*: $|A| = |A^*|$. Let's begin with the *amplitude*, which shall be denoted by ϕ. As it's seen from the above obtained expressions, the magnitude of $|A|$ coincides with the square root of the energy density in any F-adapted coordinate system. As we noted in (1.3.1) this is the sense of the quantity amplitude. So, we define it by the module of $|A| = |A^*|$. We give now two more coordinate-free ways to define the amplitude.

Recall first, that at every point, where the field is different from zero, we have three bases: the pseudoopthonormal coordinate basis $(dx, dy, dz, d\xi)$, the pseudoorthonormal basis $\chi^0 = (\mathbf{A}, \varepsilon\mathbf{A}^*, \mathbf{R}, \mathbf{S})$ and the pseudoorthogonal basis $\chi = (A, \varepsilon A^*, \mathbf{R}, \mathbf{S})$. The matrix $\chi_{\mu\nu}$ of χ with respect to the coordinate basis is

$$\chi_{\mu\nu} = \begin{Vmatrix} u & -p & 0 & 0 \\ p & u & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

We define now the amplitude ϕ of the field by

$$\phi = \sqrt{|det(\chi_{\mu\nu})|}. \tag{2.21}$$

We consider now the matrix $\mathcal{R}$ of 2-forms

$$\mathcal{R} = \begin{Vmatrix} udx \wedge d\xi & -pdx \wedge d\xi & 0 & 0 \\ pdy \wedge d\xi & udy \wedge d\xi & 0 & 0 \\ 0 & 0 & -dy \wedge dz & 0 \\ 0 & 0 & 0 & dz \wedge d\xi \end{Vmatrix},$$

or, equivalently:

$$\mathcal{R} = udx \wedge d\xi \otimes (dx \otimes dx) - pdx \wedge d\xi \otimes (dx \otimes dy) + pdx \wedge d\xi \otimes (dy \otimes dx) +$$

$$+ udy \wedge d\xi \otimes (dy \otimes dy) - dy \wedge dz \otimes (dz \otimes dz) + dz \wedge d\xi \otimes (d\xi \otimes d\xi).$$

Now we can write

$$\phi = \sqrt{\frac{1}{2}\,|R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}|}.$$

We proceed further to define the *phase* of the nonlinear solution F. We shall need the matrix $\chi^0_{\mu\nu}$ of the basis χ^0 with respect to the coordinate basis. We obtain

$$\chi^0_{\mu\nu} = \begin{Vmatrix} \frac{u}{\sqrt{u^2+p^2}} & \frac{-p}{\sqrt{u^2+p^2}} & 0 & 0 \\ \frac{p}{\sqrt{u^2+p^2}} & \frac{u}{\sqrt{u^2+p^2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

The *trace* of this matrix is

$$tr(\chi^0_{\mu\nu}) = \frac{2u}{\sqrt{u^2+p^2}}.$$

Obviously, the inequality $|\frac{1}{2}tr(\chi^0_{\mu\nu})| \leq 1$ is fulfilled. Now, by definition, the quantity $\varphi = \frac{1}{2}tr(\chi^0_{\mu\nu})$ will be called *phase function* of the solution, and the quantity

$$\theta = arccos(\varphi) = arccos\left(\frac{1}{2}tr(\chi^0_{\mu\nu})\right) \tag{2.22}$$

will be called *phase* of the solution.

Making use of the amplitude ϕ and the phase function φ we can write

$$u = \phi.\varphi, \;\; p = \phi.\sqrt{1-\varphi^2}. \tag{2.23}$$

We note that the couple of 1-forms $A = udx + pdy,\ A^* = -pdx + udy$ defines a completely integrable Pfaff system, i.e. the following equations hold:

$$\mathbf{d}A \wedge A \wedge A^* = 0,\ \mathbf{d}A^* \wedge A \wedge A^* = 0.$$

In fact, $A \wedge A^* = (u^2+p^2)dx \wedge dy$, and in every term of $\mathbf{d}A$ and $\mathbf{d}A^*$ at least one of the basis vectors dx and dy will participate, so the above exteriour products will vanish.
Remark. These considerations stay in force also for those linear solutions, which have zero invariants $I_1 = I_2 = 0$. But Maxwell's equations require u and p to be *running waves*, so the corresponding phase functions will be also running waves. As we'll see further, the phase functions for nonlinear solutions are arbitrary bounded functions.

We proceed further to define the new and important concept of *scale factor* L for a given nonlinear solution. It is defined by

$$L = \frac{|A|}{|\delta F|} = \frac{|A^*|}{|\delta * F|}. \tag{2.24}$$

Clearly, L can not be defined for the linear solutions, and in this sense it is *new* and we shall see that it is really *important*.

From the expressions $F = A \wedge \mathbf{V}$ and $*F = A^* \wedge \mathbf{V}$ it follows that the physical dimension of A and A^* is the same as that of F. We conclude that the physical dimension of L coincides with the dimension of the coordinates, i.e. $[L] = length$. From the definition it is seen that L is an *invariant* quantity, and depends on the point, in general. The invariance of L allows to define a time-like 1-form (or vector field) $f(L)\mathbf{S}$, where f is some real function of L. So, every nonlinear solution determines a time-like vector field on M.

If the scale factor L, defined by the nonlinear solution F, is a *finite* and *constant* quantity, we can introduce a *characteristic* finite time-interval $T(F)$ by the relation

$$cT(F) = L(F),$$

as well, as corresponding *characteristic frequency* by

$$\nu(F) = 1/T(F).$$

In these "wave" terms the scale factor L acquires the sense of "wave length", but this interpretation is arbitrary and we shall not make use of it.

It is clear, that the subclass of nonlinear solutions, which define constant scale factors, factors over the admissible values of the invariant $f(L)$. This

makes possible to compare with the experiment. For example, at constant scale factor L if we choose $f(L) = L/c$, then the scalar product of $(L/c)\mathbf{S}$ with the integral energy-momentum vector, which in the F-adapted coordinate system is $(0, 0, \varepsilon W, W)$, gives the invariant quantity $W.T$, having the physical dimension of action, and its numerical value could be easily measured.

2.3 Nonlinear Solutions. Description of photon-like objects

2.3.1 Explicit solutions in canonical coordinates

As it was shown in the preceding section with every nonlinear solution F of our nonlinear equations a class of F-adapted coordinate systems is associated, such that F and $*F$ acquire the form respectively

$$F = \varepsilon u dx \wedge dz + u dx \wedge d\xi + \varepsilon p dy \wedge dz + p dy \wedge d\xi$$

$$*F = -p dx \wedge dz - \varepsilon p dx \wedge d\xi + u dy \wedge dz + \varepsilon u dy \wedge d\xi .$$

After some elementary calculations we obtain

$$\delta F = (u_\xi - \varepsilon u_z)dx + (p_\xi - \varepsilon p_z)dy + \varepsilon(u_x + p_y)dz + (u_x + p_y)d\xi ,$$

$$\delta * F = -\varepsilon(p_\xi - \varepsilon p_z)dx + \varepsilon(u_\xi - \varepsilon p_z)dy - (p_x - u_y)dz - (p_x - u_y)d\xi ,$$

$$F_{\mu\nu}(\delta F)^\nu dx^\nu = (*F)_{\mu\nu}(\delta * F)^\nu dx^\nu =$$

$$= \varepsilon \left[p(p_\xi - \varepsilon p_z) + u(u_\xi - \varepsilon u_z)\right] dz + \left[p(p_\xi - \varepsilon p_z) + u(u_\xi - \varepsilon u_z)\right] d\xi ,$$

$$(\delta F)^2 = (\delta * F)^2 = -(u_\xi - \varepsilon u_z)^2 - (p_\xi - \varepsilon p_z)^2$$

A simple direct calculation shows, that the equation

$$F_{\mu\nu}(\delta * F)^\nu + (*F)_{\mu\nu}(\delta F)^\nu = 0$$

is identically fulfilled for any such F with arbitrary u and p. We obtain that our equations reduce to only 1 equation, namely

$$p(p_\xi - \varepsilon p_z) + u(u_\xi - \varepsilon u_z) = \frac{1}{2}\left[(u^2 + p^2)_\xi - \varepsilon(u^2 + p^2)_z\right] = 0. \qquad (2.25)$$

The obvious solution of this equation is

$$u^2 + p^2 = \phi^2(x, y, \xi + \varepsilon z). \tag{2.26}$$

The solution obtained shows that the equations impose some limitations only on the amplitude function ϕ and the phase function φ is arbitrary exept that it is bounded: $|\varphi| \leq 1$. The amplitude ϕ is a running wave along the specially chosen coordinate z, which is common for all F-adapted coordinate systems.Considered as a function of the spatial coordinates, the amplitude ϕ is *arbitrary*, so it can be chosen *spatially finite*. The time-evolution does not affect the initial form of ϕ, so it will stay the same in time. This shows, that *among the nonlinear solutions of our equations there are (3+1) soliton-like solutions*. The spatial structure is determined by the initial condition, while the phase function φ can be used to define *internal dynamics* of the solution.

Recalling the substitutions (2.23)

$$u = \phi.\varphi, \; p = \phi\sqrt{1 - \varphi^2},$$

and the equality $|A| = \phi$, we get

$$|\delta F| = |\delta * F| = \frac{|\phi||\varphi_\xi - \varepsilon\varphi_z|}{\sqrt{1 - \varphi^2}}, \; L = \frac{\sqrt{1 - \varphi^2}}{|\varphi_\xi - \varepsilon\varphi_z|}. \tag{2.27}$$

For the induced pseudoorthonormal bases (1-forms and vector fields) we find

$$\mathbf{A} = \varphi dx + \sqrt{1 - \varphi^2} dy, \; \varepsilon\mathbf{A}^* = -\sqrt{1 - \varphi^2} dx + \varphi dy, \; \mathbf{R} = -dz, \; \mathbf{S} = d\xi,$$

$$\mathbf{A} = -\varphi\frac{\partial}{\partial x} - \sqrt{1 - \varphi^2}\frac{\partial}{\partial y}, \; \varepsilon\mathbf{A}^* = \sqrt{1 - \varphi^2}\frac{\partial}{\partial x} - \varphi\frac{\partial}{\partial y}, \; \mathbf{R} = \frac{\partial}{\partial z}, \; \mathbf{S} = \frac{\partial}{\partial \xi}.$$

Hence, the nonlinear solutions in canonical coordinates are parametrized by one function ϕ of 3 parameters and one *bounded* function of 4 parameters. Therefore, the separation of various subclasses of nonlinear solutions is made by imposing additional conditions on these two functions. Further in this subsection we are going to separate a subclass of solutions , the integral properties of which reflect well enough the well known from the experiment integral properties and characteristics of the free photons. These solutions will be called *photon-like* and will be separated through imposing additional requirements on φ and L in a coordinate-free manner.

We note first, that we have three invariant quantities at hand: ϕ, φ and L. The amplitude function ϕ is to be determined by the initial conditions, which have to be *finite*. So, we may impose additional conditions on L and φ. These conditions have to express some intra-consistency among the various characteristics of the solution. The idea, what kind of intra-consistency to use, comes from the observation that the amplitude function ϕ is a first integral of the vector field $\mathbf{V}$, i.e.

$$\mathbf{V}(\phi) = \left(-\varepsilon\frac{\partial}{\partial z} + \frac{\partial}{\partial \xi}\right)(\phi) = -\varepsilon\frac{\partial}{\partial z}\phi(x, y, \xi + \varepsilon z) + \frac{\partial}{\partial \xi}\phi(x, y, \xi + \varepsilon z) = 0.$$

We want to extend this available consistency between $\mathbf{V}$ and ϕ, i.e. we shall require the two functions φ and L to be first integrals of some of the available vector fields. Explicitly, we require the following:

1^0. *The phase function φ is a first integral of the three vector fields $\mathbf{A}, \mathbf{A}^*$ and $\mathbf{R}$:* $\mathbf{A}(\varphi) = 0, \mathbf{A}^*(\varphi) = 0, \mathbf{R}(\varphi) = 0$.

2^0. *The scale factor L is a non-zero finite first integral of the vector field $\mathbf{S}$:* $\mathbf{S}(L) = 0$.

The requirement $\mathbf{R}(\varphi) = 0$ just means that in these coordinates φ does not depend on the coordinate z. The two other equations of 1^0 define the following system of differential equations for φ:

$$-\varphi\frac{\partial \varphi}{\partial x} - \sqrt{1-\varphi^2}\frac{\partial \varphi}{\partial y} = 0,\ \ \sqrt{1-\varphi^2}\frac{\partial \varphi}{\partial x} - \varphi\frac{\partial \varphi}{\partial y} = 0.$$

Noticing that the matrix

$$\left\| \begin{matrix} -\varphi & -\sqrt{1-\varphi^2} \\ \sqrt{1-\varphi^2} & -\varphi \end{matrix} \right\|$$

has non-zero determinant, we conclude that the only solution of the above system is the zero-solution:

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial y} = 0.$$

We obtain that in the coordinates used the phase function φ depends only on ξ. Therefore, in view of (2.27), for the scale factor L we get

$$L = \frac{\sqrt{1-\varphi^2}}{|\varphi_\xi|}.$$

Now, the requirement 2^0, which in these coordinates reads

$$\frac{\partial L}{\partial \xi} = \frac{\partial}{\partial \xi} \frac{\sqrt{1-\varphi^2}}{|\varphi_\xi|} = 0,$$

just means that the scale factor L is a pure constant: $L = const$. In this way we obtain the differential equation

$$\frac{\partial \varphi}{\partial \xi} = \mp \frac{1}{L}\sqrt{1-\varphi^2}. \tag{2.28}$$

The obvious solution to this equation reads

$$\varphi(\xi) = cos\left(\kappa \frac{\xi}{L} + const\right), \tag{2.29}$$

where $\kappa = \pm 1$. It is worth to note that the characteristic *frequency*

$$\nu = \frac{c}{L} \tag{2.30}$$

has nothing to do with the frequency in CED. In fact, the quantity L can not be defined in Maxwell's theory.

Finally we note, that the so obtained phase function $\varphi(\xi)$ leads to the following. The 2-form $tr(\mathcal{R}^0)$, where $\mathcal{R}^0$ is the matrix of 2-forms, formed similarly to the matrix $\mathcal{R}$, but using the basis $(\mathbf{A}, \varepsilon\mathbf{A}^*, \mathbf{R}, \mathbf{S})$ instead of the basis $(A, \varepsilon A^*, \mathbf{R}, \mathbf{S})$, is closed. In fact,

$$tr(\mathcal{R}^0) = \varphi dx \wedge d\xi + \varphi dy \wedge d\xi - dy \wedge dz + dz \wedge d\xi$$

and since $\varphi = \varphi(\xi)$, we get $\mathbf{d}tr(\mathcal{R}^0) = 0$. Note also that the above explicit form of $tr(\mathcal{R}^0)$ allows to define the phase function by

$$\varphi = \sqrt{\frac{|tr(\mathcal{R}^0)|^2}{2}}.$$

Remark. If one of the two functions u and p, for example p, is equal to zero: $p = 0$, then formally we again have a solution, which may be called *linearly polarized* by obvious reasons. Clearly, the phase function of such solutions will be *constant*: $\varphi = const$, so, the corresponding scale factor becomes infinitely large: $L \to \infty$, therefore, condition 2^0 is not satisfied. The reason for this is, that at $p = 0$ the function u becomes a *running wave* and we get $|\delta F| = |\delta * F| = 0$, so the scale factor can not be defined by (2.24).

2.3.2 Intrinsic angular momentum (spin, helicity)

The problem for describing the *intrinsic angular momentum* (IAM), or in short *helicity, spin* of the photon is of fundamental importance in modern physics, therefore, we shall pay a special attention to it. In particular, we are going to consider two approaches for its mathematical description. But first, some preceding comments.

First of all, *there is no any doubt that every free photon carries such an intrinsic angular momentum.* Since the angular momentum is a conserved quantity, the existence of the photon's intrinsic angular momentum can be easily established and, in fact, its presence has been experimentally proved by an immediate observation of its mechanical action and its value has been numerically measured. Assuming this is so, we have to understand its origin, nature and its entire meaning for the existence and outer relations of those natural entities, called shortly photons somewhere in the first quarter of this century.

So, we begin with the assumption:*every free photon carries an intrinsic angular momentum with integral value equal to the Planck's constant h.* According to our understanding, the photon's IAM comes from an intrinsic *periodic process.* This point of view undoubtedly leads to the notion, that photons *are not* point-like structureless objects, they have a structure, i.e. they are *extended objects.* In fact, according to one of the basic principles of physics *all free objects move as a whole uniformly.* So, if the photon is a point-like object any characteristic of a periodic process, e.g. frequency, should come from an outside force field, i.e. it can not be free: a free point-like (structureless) object can not have the characteristic frequency.

This simple, but true, conclusion sets the theoretical physics of the first quarter of this century faced with a serious dilemma: to keep the notion of structurelessness and to associate in a formal way the characteristic frequency to the microobjects, or to leave off the notion of structuelesness, to assume the notion of extendedness and availability of intrinsically occurring periodic process and to build corresponding integral characteristics, determined by this periodic process. A look back in time shows that the majority of those days physicists had adopted the first approach, which has brought up to life quantum mechanics as a computing method , and the dualistic-probabilistic interpretation as a philosophical conception. If we set aside the wide spread and intrinsically controversial idea that all microobjects are at the same time (point-like) particles and (infinite) waves, and look impartially, in a fair-

minded way, at the quantum mechanical wave function for a *free* particle, we see that *the only positive consequence* of its introduction is *the legalization of frequency*, as an inherent characteristic of the microobject. In fact, the probabilistic interpretation of the quantum mechanical wave function for a free object, obtained as a solution of the free Schroedinger equation, is impossible since its square is not an integrable quantity (the integral is infinite). The frequency is really needed not because of the dualistic-probabilistic nature of microobjects, it is needed because the Planck's relation $E = h\nu$ turns out to be universally true in microphysics, so there is no way to avoid the introduction of frequency. The question is, if the introduction of frequency necessarily requires some (linear) "wave equation" and the simple complex exponentials of the kind $const.exp[i(\mathbf{k}.\mathbf{r} - \nu t)]$, i.e. running waves, as "free solutions". Our answer to this question is "no". The classical wave is something much richer and much more engaging concept, so it hardly worths to use it just because of the attribute of frequency. In our opinion, it suffices to have a periodic process at hand.

These considerations made us turn to the soliton-like objects, they realize the two features of the microobjects (*localized spatial extendedness and time-periodicity*), simultaneously, and, therefore, seem to be more adequate theoretical models for those microobjects, obeying the Planck's relation $E = h\nu$. Of course, if we are interested only in the behaviour of the microobject as a whole, we can use the point-like notion, but any attempt to give a meaning of its integral characteristics without looking for their origin in the consistent intrinsic dynamics and structure, in our opinion, is not a perspective theoretical idea. And the "stumbling point" of such an approach is just the availability of an intrinsic mechanical angular momentum, which can not be understood as an attribute of a free structureless object.

Having in view the above considerations, we are going to consider two ways to introduce and define the intrinsic angular momentum as a local quantity and to obtain, by integration, its integral value. So, these two approaches will be of use only for the spatially finite nonlinear solutions of our equations. The both approaches introduce in different ways 3-tensors (2-covariant and 1-contravariant). Although these two 3-tensors are built of quantities, connected in a definite way with the field F, their nature is quite different. The first approach is based on an appropriate tensor generalization of the classical Poynting vector. The second approach makes use of the concept of *torsion*, connected with the field F, considered as 1-covariant and

1-contravariant tensor. The first approach is pure algebraic, while the second one uses derivatives of $F_{\mu\nu}$. The spatially finite nature of the solutions F allows to build corresponding integral conserved quantities, naturally interpreted as angular momentum. The scale factor L appears as a multiple, so these quantities go to infinity for all linear (i.e. for Maxwell's) solutions.

In the first approach we make use of the scale factor L, the isotropic vector field $\mathbf{V}$ and the two 1-forms A and A^*. By these four quantities we build the following 3-tensor H:

$$H = \kappa\frac{L}{c}\mathbf{V} \otimes (A \wedge A^*). \tag{2.31}$$

The connection with the classical vector of Poynting comes through the exteriour product of A and A^*, the 3-dimensional sense of which is just the Pointing's vector. In components we have

$$H^\mu_{\nu\sigma} = \kappa\frac{L}{c}\mathbf{V}^\mu(A_\nu A^*_\sigma - A_\sigma A^*_\nu).$$

In our system of coordinates we get

$$H = \kappa\frac{L}{c}\left(-\varepsilon\frac{\partial}{\partial z} + \frac{\partial}{\partial \xi}\right) \otimes (\varepsilon\phi^2 dx \wedge dy),$$

so, the only non-zero components are

$$H^3_{12} = -H^3_{21} = -\kappa\frac{L}{c}\phi^2,\ H^4_{12} = -H^4_{21} = \kappa\varepsilon\frac{L}{c}\phi^2.$$

It is easily seen, that the divergence $\nabla_\mu H^\mu_{\nu\sigma} \to \nabla_\mu H^\mu_{12}$ is equal to 0. In fact,

$$\nabla_\mu H^\mu_{12} = \frac{\partial}{\partial z}H^3_{12} + \frac{\partial}{\partial \xi}H^4_{12} = \kappa\frac{L}{c}\left[-(\phi^2)_z + (\varepsilon\phi^2)_\xi\right] = 0$$

because ϕ^2 is a running wave along the coordinate z. Since the tangent bundle is trivial we may construct the antisymmetric 2-tensor

$$\mathbf{H}_{\nu\sigma} = \int_{R^3} H_{4,\nu\sigma}dxdydz,$$

the constant components of which are conserved quantities.

$$\mathbf{H}_{12} = -\mathbf{H}_{21} = \int_{R^3} H_{4,12}dxdydz = \kappa\varepsilon\frac{L}{c}W = \kappa\varepsilon WT = \kappa\varepsilon\frac{W}{\nu}.$$

The non-zero eigen values of $\mathbf{H}_{\nu\sigma}$ are pure imaginary and are equal to $\pm iWT$. This tensor has unique non-zero invariant $P(F)$,

$$P(F)=\sqrt{\frac{1}{2}\mathbf{H}_{\nu\sigma}\mathbf{H}^{\nu\sigma}}=WT. \tag{2.32}$$

The quantity $P(F)$ will be called *Planck's invariant* for the finite nonlinear solution F. All finite nonlinear solutions $F_1, F_2, ...$, satisfying the condition

$$P(F_1)=P(F_2)=...=h,$$

where h is the Planck's constant, will be called further *photon-like*. The tensor field H will be called *intrinsic angular momentum tensor* and the tensor $\mathbf{H}$ will be called *spin tensor* or *helicity tensor*. The Planck's invariant $P(F)=WT$, having the physical dimension of action, will be called *integral angular momentum*, or just *spin* or *helicity*.

The reasons to use this terminology are quite clear: the time evolution of the two mutually orthogonal vector fields A and A^* is a rotational-advancing motion around and along the z-coordinate (admissible are the right and the left rotations: $\kappa=\pm 1$) with the advancing velocity of c and the frequency of circulation $\nu=c/L$. We see the basic role of the two features of the solutions: their soliton-like character, giving finite value of all integral quantities, and their nonlinear character, allowing to define the scale factor L correctly. From this point of view the intrinsic angular momentum h of a free photon is far from being *incomprihensible* quantity, connected with the even more incomprihensible duality "wave-particle", and it looks as a quite normal integral characteristic of a solution, presenting a model of our knowledge of the free photon.

We proceed to the second approach by recalling the definition of *torsion* of two (1,1) tensors. If G and K are 2 such tensors

$$G=G^\nu_\mu dx^\mu\otimes\frac{\partial}{\partial x^\nu}, \quad K=K^\nu_\mu dx^\mu\otimes\frac{\partial}{\partial x^\nu},$$

their torsion is defined as a 3-tensor $S^\sigma_{\mu\nu}=-S^\sigma_{\nu\mu}$ by the equation

$$S(G,K)(X,Y)=[GX,KY]+[KX,GY]+GK[X,Y]+KG[X,Y]-$$

$$-G[X,KY]-G[KX,Y]-K[X,GY]-K[GX,Y],$$

where $[,]$ is the Lie-bracket of vector fields,

$$GX = G^\nu_\mu X^\mu \frac{\partial}{\partial x^\nu}, \quad GK = G^\nu_\mu K^\mu_\sigma dx^\sigma \otimes \frac{\partial}{\partial x^\nu}$$

and X, Y are 2 arbitrary vector fields. If $G = K$, in general $S(G,G) \neq 0$ and

$$S(G,G)(X,Y) = 2\left\{[GX,GY] + GG[X,Y] - G[X,GY] - G[GX,Y]\right\}.$$

This last expression defines at every point $x \in M$ the torsion $S(G,G) = S_G$ of G with respect to the 2-dimensional plane, defined by the two vectors $X(x)$ and $Y(x)$. Now we are going to compute the torsion S_F of the nonlinear solution F with respect to the intrinsically defined by the two unit vectors $\mathbf{A}$ and $\varepsilon\mathbf{A}^*$ 2-plane. In components we have

$$(S_F)^\sigma_{\mu\nu} = 2\left[F^\alpha_\mu \frac{\partial F^\sigma_\nu}{\partial x^\alpha} - F^\alpha_\nu \frac{\partial F^\sigma_\mu}{\partial x^\alpha} - F^\sigma_\alpha \frac{\partial F^\alpha_\nu}{\partial x^\mu} + F^\sigma_\alpha \frac{\partial F^\alpha_\mu}{\partial x^\nu}\right].$$

In our coordinate system

$$\mathbf{A} = -\varphi\frac{\partial}{\partial x} - \sqrt{1-\varphi^2}\frac{\partial}{\partial y}, \quad \varepsilon\mathbf{A}^* = \sqrt{1-\varphi^2}\frac{\partial}{\partial x} - \varphi\frac{\partial}{\partial y},$$

so,

$$(S_F)^\sigma_{\mu\nu}\mathbf{A}^\mu\varepsilon\mathbf{A}^{*\nu} = (S_F)^\sigma_{12}(\mathbf{A}^1\varepsilon\mathbf{A}^{*2} - \mathbf{A}^2\varepsilon\mathbf{A}^{*1}).$$

For $(S_F)^\sigma_{12}$ we get

$$(S_F)^1_{12} = (S_F)^2_{12} = 0, \quad (S_F)^3_{12} = -\varepsilon(S_F)^4_{12} = 2\varepsilon\{p(u_\xi - \varepsilon u_z) - u(p_\xi - \varepsilon p_z)\}.$$

Remark. In our case $(S_F)^\sigma_{12} = (S_{*F})^\sigma_{12}$, so further we shall work with S_F only.

It is easily seen that the following relation holds: $\mathbf{A}^1\varepsilon\mathbf{A}^{*2} - \mathbf{A}^2\varepsilon\mathbf{A}^{*1} = 1$. Now, for the case

$$u = \phi(x,y,\xi+\varepsilon z)\cos\left(\kappa\frac{\xi}{L} + const\right), \quad p = \phi(x,y,\xi+\varepsilon z)\sin\left(\kappa\frac{\xi}{L} + const\right)$$

we obtain

$$(S_F)^3_{12} = -\varepsilon(S_F)^4_{12} = -2\varepsilon\frac{\kappa}{L}\phi^2,$$

$$(S_F)^{\sigma}_{\mu\nu}\mathbf{A}^{\mu}\varepsilon\mathbf{A}^{*\nu} = \left[0, 0, -2\varepsilon\frac{\kappa}{L}\phi^2, 2\frac{\kappa}{L}\phi^2\right].$$

Since ϕ^2 is a running wave along the z-coordinate, the vector field $S_F(\mathbf{A}, \varepsilon\mathbf{A}^*)$ has zero divergence: $\nabla_\nu \left[S_F(\mathbf{A}, \varepsilon\mathbf{A}^*)\right]^\nu = 0$. Now we define the *helicity vector* for the solution F by

$$\Sigma_F = \frac{L^2}{2c}S_F(\mathbf{A}, \varepsilon\mathbf{A}^*).$$

Since $L = const$, then Σ_F has also zero divergence, so the integral quantity

$$\int (\Sigma_F)_4 dxdydz$$

does not depend on time and is equal to κWT. The photon-like solutions are separated in the same way by the condition $WT = h$. Here are three more integral expressions for the quantity WT. We form the 4-form

$$-\frac{1}{L}\mathbf{S} \wedge *\Sigma_F = \frac{\kappa}{c}\phi^2\omega_\circ$$

and integrate it over the 4-volume $\mathcal{R}^3 \times L$, the result is κWT. Besides, we verify easily the relations

$$\frac{1}{c}\int_{R^3\times L} |A \wedge A^*|\omega_\circ = \frac{L^2}{c}\int_{R^3\times L} |\delta F \wedge \delta * F|\omega_\circ = WT.$$

Since we separate the photon-like solutions by the relation $WT = h$, the last expressions suggest the following interpretation of the Planck's constant h. Since $|A \wedge A^*|$ is proportional to the area of the square, defined by the two mutually orthogonal vectors A and εA^*, the above integral sums up all these areas over the whole 4-volume, occupied by the solution F during the intrinsically determined time period T, in which the couple $(A, \varepsilon A^*)$ completes a full rotation. The same can be said for the couple $(\delta F, \delta * F)$ with some different factor in front of the integral. This shows quite clearly the "helical" origin of the full energy $W = h\nu$ of the single photon.

2.3.3 Solutions in spherical cordinates

The so far obtained soliton-like solutions describe objects, "coming from infinity" and "going to infinity". Of interest are also soliton like solutions

"radiated" from, or "absorbed", by some central "source" and propagating radially from or to the center of this source. We are going to show, that our equations admit such solutions too. We assume this central source to be a small ball R^0 with radius $r_\circ$, and put the origin of the coordinate system at the center of the source-ball. The standard spherical coordinates $(r, \theta, \varphi, \xi)$ will be used and all considerations will be carried out in the region out of the ball R^0. In these coordinates we have

$$ds^2 = -dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\varphi^2 + d\xi^2, \ \sqrt{|\eta|} = r^2 sin\theta.$$

The $*$-operator acts in these coordinates as follows:

$$\begin{array}{ll}
*dr = r^2 sin\theta d\theta \wedge d\varphi \wedge d\xi & *(dr \wedge d\theta \wedge d\varphi) = (r^2 sin\theta)^{-1} d\xi \\
*d\theta = -sin\theta dr \wedge d\varphi \wedge d\xi & *(dr \wedge d\theta \wedge d\xi) = sin\theta d\varphi \\
*d\varphi = (sin\theta)^{-1} dr \wedge d\theta d\xi & *(dr \wedge d\varphi \wedge d\xi) = -(sin\theta)^{-1} d\theta \\
*d\xi = r^2 sin\theta dr \wedge d\theta d\varphi & *(d\theta \wedge d\varphi \wedge d\xi) = (r^2 sin\theta)^{-1} dr
\end{array}$$

$$\begin{array}{ll}
*(dr \wedge d\theta) = -sin\theta d\varphi \wedge d\xi & *(d\theta \wedge d\varphi) = -(r^2 sin\theta)^{-1} dr \wedge d\xi \\
*(dr \wedge d\varphi) = (sin\theta)^{-1} d\theta \wedge d\xi & *(d\theta \wedge d\xi) = -sin\theta dr \wedge d\varphi \\
*(dr \wedge d\xi) = r^2 sin\theta d\theta \wedge d\varphi & *(d\varphi \wedge d\xi = (sin\theta)^{-1} dr \wedge d\theta.
\end{array}$$

We look for solutions of the following kind:

$$F = \varepsilon u dr \wedge d\theta + u d\theta \wedge d\xi + \varepsilon p dr \wedge d\varphi + p d\varphi \wedge d\xi, \tag{2.33}$$

where u and p are spatially finite functions. We get

$$*F = \frac{p}{sin\theta} dr \wedge d\theta + \varepsilon \frac{p}{sin\theta} d\theta \wedge d\xi - u sin\theta dr \wedge d\varphi - \varepsilon sin\theta d\varphi \wedge d\xi.$$

The following relations hold:

$$F \wedge F = 2\varepsilon(up - up) dr \wedge d\theta \wedge d\varphi \wedge d\xi = 0,$$

$$F \wedge *F = \left(-u^2 sin\theta + u^2 sin\theta - \frac{p^2}{sin\theta} + \frac{p^2}{sin\theta}\right) dr \wedge d\theta \wedge d\varphi \wedge d\xi = 0,$$

i.e. the two invariants are equal to zero: $(*F)_{\mu\nu} F^{\mu\nu} = 0, \ F_{\mu\nu} F^{\mu\nu} = 0$.

After some elementary computation we obtain

$$\delta F \wedge F = \delta * F \wedge *F = \varepsilon \left[u\left(\varepsilon p_r + p_\xi\right) - p\left(\varepsilon u_r + u_\xi\right)\right] dr \wedge d\theta \wedge d\varphi +$$

$$+ \left[u\left(\varepsilon u_r + u_\xi\right) - u\left(\varepsilon p_r + p_\xi\right)\right] d\theta \wedge d\varphi \wedge d\xi,$$

$$F\wedge *\mathbf{d}F=\varepsilon\left[u\left(\varepsilon u_r+u_\xi\right)sin\theta+\frac{p\left(\varepsilon p_r+p_\xi\right)}{sin\theta}\right]dr\wedge d\theta\wedge d\varphi-$$

$$-\varepsilon\left[u\left(\varepsilon u_r+u_\xi\right)sin\theta+\frac{p\left(\varepsilon p_r+p_\xi\right)}{sin\theta}\right]d\theta\wedge d\phi\wedge d\xi,$$

$$(*F)\wedge *\mathbf{d}*F=\left[u\left(\varepsilon u_r+u_\xi\right)sin\theta+\frac{p\left(\varepsilon p_r+p_\xi\right)}{sin\theta}\right]dr\wedge d\theta\wedge d\varphi-$$

$$-\left[u\left(\varepsilon u_r+u_\xi\right)sin\theta+\frac{p\left(\varepsilon p_r+p_\xi\right)}{sin\theta}\right]d\theta\wedge d\phi\wedge d\xi.$$

So, the two functions u and p have to satisfy the equation

$$u\left(\varepsilon u_r+u_\xi\right)sin\theta+\frac{p\left(\varepsilon p_r+p_\xi\right)}{sin\theta}=0, \tag{2.34}$$

which is equivalent to the equation

$$\left(u^2 sin\theta+\frac{p^2}{sin\theta}\right)_\xi+\varepsilon\left(u^2 sin\theta+\frac{p^2}{sin\theta}\right)_r=0. \tag{2.35}$$

The general solution of this equation is

$$u^2 sin\theta+\frac{p^2}{sin\theta}=\phi^2(\xi-\varepsilon r,\theta,\phi). \tag{2.36}$$

For the non-zero components of the energy-momentum tensor we obtain

$$-Q_1^1=-Q_1^4=Q_4^1=Q_4^4=\frac{1}{4\pi r^2 sin\theta}\left(u^2 sin\theta+\frac{p^2}{sin\theta}\right). \tag{2.37}$$

It is seen that the energy density is not exactly a running wave but when we integrate to get the integral energy, the integrand is exactly a running wave:

$$W=\frac{1}{4\pi}\int_{R^3-R^0}*\left(Q_\mu^4 d\xi\right)=\frac{1}{4\pi}\int_{R^3-R^0}\left(u^2 sin\theta+\frac{p^2}{sin\theta}\right)dr\wedge d\theta\wedge d\phi.$$

Since the functions u and p are spatially finite, the integral energy W is finite, and from the explicit form of the energy-momentum tensor it follows the well known relation between the integral energy and momentum: $W^2-c^2\mathbf{p}^2=0$.

2.4 Interference, Nonlinearity and Superposition

2.4.1 Introductory remarks

Having at hand the photon-like solutions a natural next step is to try to describe the situation when two photons occupy the same (or partially the same) 3-region in some period of time. It is clear, that if these two photons meet somewhere, i.e. their cilinder-like world-tubes intersect non-trivially, the interesting case is when they *move along the same spatial straight line* and in *the same direction*. Since they move by the same velocities they will continue to overlap each other until some outer agent causes a change. What kind of an object is obtained in this way, is it a photon or not, what kind of interaction takes place, what is its integral energy , its momentum and its angular momentum? Many challenging and still not answered questions may be set in this direction before the theoretical physics. And this section is devoted to consideration of some of these problems in the frame of Extended electrodynamics.

Almost all experiments set to find some immediate mutual interaction of two (or more) electromagnetic fields in vacuum, causing some observable effects (e.g. frequency or amplitude changes), as far as we know, have faild, exept when the two fields satisfy the so called *coherence conditions*. In the frame of CED and working with plane waves this simply means, that their *phase difference must be a constant quantity*. The usual way of consideration is limited to *cosine*-like running waves with the *same* frequency. The physical explanation is based on the linearity of Maxwell's equations, which require any linear combination of solutions to be again a solution, so the "building points" of the medium, subject to the field pressure of the two independent fields, go out of their equilibrium state obeying simoultaneously the two forces applied in the overlaping 3-region. After getting out of this overlaping 3-region the fields stay what they have been before the interaction. In order to describe the interaction, i.e. the observed redistribution of the energy-momentum density inside the overlaping 3-region, CED uses the corresponding mathematical expressions in Maxwell's theory and gets comparetively good results. Most frequently the Poynting vector $S \sim [(E_1 + E_2) \times (B_1 + B_2)]$ is used and the cross-terms $(E_1 \times B_2) + (E_2 \times B_1)$ are held responsible for the interaction, in fact, the very *interference* is de-

fined by the condition that these cross-terms, usually called "interference terms", are different from zero. But, the interference takes really place *only when the coherence conditions are met*, while CED permits interference almost always. In other words, from Maxell's theory we can not obtain these coherence conditions as *necessary conditions* for some interference to take place.

In EED we *have no superposition principle*, so we have to approach this physical situation in a new way. First, let's specify the situation more in detail and in terms of the notion for EM-field in EED. Roughly speaking, this notion is based on the idea for discreteness, i.e. the real electromagnetic fields consist of many noninteracting, or very weakly interacting, photons, moving in *various* directions. Because of the great velocity of their straight line motion it is hardly possible to observe and say what happens when two photons meet somewhere. The experiment shows that in the most cases they pass through each other and forget about the meeting. As we mentioned above, the interseting case is when they move along the same direction and the regions, they occupy, overlap nontrivially.

The nonlinear solutions we obtained in the preceding section can not describe such set of photons, moving in *various* directions. Even if we choose the amplitude function ϕ to consist of many "3-bubles" (since ϕ is a running wave, all these bubles, are parts of the same running wave), they all have to move in the same direction, which is a special, but not the general, case of the situation we consider here. So, in order to incorporate for description such situations, some perfection of EED is needed. As before, this perfection shall consist of two steps: first, elaboration of the algebraic character of the mathematical field, second, elaboration of the equations. The second step, besides its dynamical task, must define also the necessary conditions for interference of photon-like solutions, which should coincide with the above mentioned, experimentally established and repeatedly confirmed coherence conditions. We shall see that this is easily achievable in EED.

2.4.2 Elaborating the mathematical object

Recall that our mathematical object that represents the field is a 2-form with values in $\mathcal{R}^2$. We want to elaborate it in order to reflect more fully the physical situation. The new moment is that inside the 3-region under consideration we have *many* photons. Each of these photons, considered as independent objects, is described by a pfoton-like solution as given in the

preceding section, i.e. each of them has its own spatial structure, its own scale factor (or frequency) and its own direction of motion as a whole. Of course, the velocity of motion is the same for all of them. To this physical situation we have to juxtapose *one* mathematical object, which have to generalize in a natural way our old object Ω. The idea for this generalization is very simple and consists in the following. With every single photon, we associate its own $\mathcal{R}^2$-space, so if the number of the presenting photons is N, we'll have N such spaces, or a new N-dimensional vector space. Denoting this vector space by $\mathcal{N}$, our object becomes a 2-form Ω with values in the vector space $\mathcal{R}^2 \otimes \mathcal{N}$: $\Omega \in \Lambda^2 (M, \mathcal{R}^2 \otimes \mathcal{N})$. We recall now how this vector space $\mathcal{N}$ is explicitly built.

If $\mathcal{K}$ is an arbitrary set, finite or infinite, we consider those mappings of this set into a given field, e.g. $\mathcal{R}$, which are different from zero only for finite number of elements of $\mathcal{K}$. These mappings will be the elements of the space $\mathcal{N}$. A basis of this space is built in the following way. We consider the elements $f \in \mathcal{N}$, having the property: if $a \in \mathcal{K}$ then $f(a) = 1$ and f has zero values for all othe elements of $\mathcal{K}$. So, with every element $a \in \mathcal{K}$ we associate the corresponding element $f_a \in \mathcal{N}$, therefore, an arbitrary element $f \in \mathcal{N}$ is represented as follows:

$$f = \sum_{i=1}^{N} \left(\lambda^i f_{a_i} \right),$$

where λ^i, $i = 1, 2, ..., N$, are the values, aquired by F, when i runs from 1 to N (of course, some of the λ's may be equal to zero). The linear structure in $\mathcal{N}$ is naturally introduced, making use of the linear structure in $\mathcal{R}$ in the well known way. The linear *independence* of f_{a_i} is easily shown. In fact, assuming the opposite, i.e. that there exist such λ^i, among which at least one is not zero and the following relation holds

$$\sum_{i=1}^{N} \lambda^i f_{a_i} = 0,$$

then for any $j = 1, 2, ..., N$ we'll have

$$\sum_{i=1}^{N} \lambda^i f_{a_i}(a_j) = \lambda^j = 0,$$

which contradicts the assumption. Hence, f_{a_i} define really a basis of $\mathcal{N}$. Now we form the injective mapping $i_N : N \to \mathcal{N}$, defined by

$$i_N(a) = f_a, \ a \in N,$$

so the set N turns into a basis of $\mathcal{N}$. If such a construction is made, then $\mathcal{N}$ is called a *free vector space over the set* N. Further onthe corresponding basis of our set of photons will be denoted by E_a. So, our mmathematical object will look as follows (summing up over the repeating index a)

$$\Omega = \Omega^a \otimes E_a = [F^a \otimes e_1^a + (*F)^a \otimes e_2^a] \otimes E_a, \tag{2.38}$$

where (e_1^a, e_2^a) is the associated with the field F^a basis. If we work in an arbitrary basis of $\mathcal{R}^2$, the full writing reads (i=1,2)

$$\Omega = \Omega^a \otimes E_a = \Omega_i^a \otimes k_a^i \otimes E_a. \tag{2.39}$$

Following the several times used already method we define the product of two 2-forms of the kind (2.39). For erxample, if $\Phi = \Phi_i^a \otimes k_a^i \otimes E_a$, $\Psi = \Psi_j^b \otimes l_b^j \otimes E_b$, we'll have

$$(\vee,\vee)(\Phi,\Psi) = (\vee,\vee)(\Phi_i^a \otimes k_a^i \otimes E_a, \Psi_j^b \otimes l_b^j \otimes E_b) =$$

$$= \sum_{a=1}^{N} \Big[\Phi_1^a \wedge \Psi_1^a \otimes k_a^1 \vee l_a^1 + \Phi_2^a \wedge \Psi_2^a \otimes k_a^2 \vee l_a^2 +$$

$$+(\Phi_1^a \wedge \Psi_2^a + \Phi_2^a \wedge \Psi_1^a) \otimes k_a^1 \vee l_a^2\Big] \otimes E_a \vee E_a +$$

$$+ \sum_{a<b=1}^{N} \Big[(\Phi_1^a \wedge \Psi_1^b + \Phi_1^b \wedge \Psi_1^a) \otimes k_a^1 \vee l_b^1 + (\Phi_2^a \wedge \Psi_2^b + \Phi_2^b \wedge \Psi_2^a) \otimes k_a^2 \vee l_b^2 +$$

$$+(\Phi_1^a \wedge \Psi_2^b + \Phi_2^a \wedge \Psi_1^b + \Phi_1^b \wedge \Psi_2^a + \Phi_2^b \wedge \Psi_1^a) \otimes k_a^1 \vee l_b^2\Big] \otimes E_a \vee E_b.$$

Let now Ω be of the kind $\Omega = (F^a \otimes e_a^1 + *F^a \otimes e_a^2) \otimes E_a$. Then, forming $*\Omega$ and $\delta\Omega$, for $(\vee,\vee)(\delta\Omega, *\Omega)$ we obtain

$$(\vee,\vee)(\delta\Omega, *\Omega) = \sum_{a=1}^{N} \Big[\delta F^a \wedge *F^a \otimes e_a^1 \vee e_a^1 + \delta * F^a \wedge ** F^a \otimes e_a^2 \vee e_a^2 +$$

$$+(\delta F^a \wedge ** F^a + \delta * F^a \wedge *F^a) \otimes e_a^1 \vee e_a^2\Big] \otimes E_a \vee E_a +$$

$$+ \sum_{a<b=1}^{N} \Big[(\delta F^a \wedge *F^b + \delta F^b \wedge *F^a) \otimes e_a^1 \vee e_b^1 + (\delta *F^a \wedge **F^b + \delta *F^b \wedge **F^a) \otimes e_a^2 \vee e_b^2$$

$$+(\delta F^a \wedge **F^b + \delta * F^a \wedge *F^b + \delta F^b \wedge **F^a + \delta * F^b \wedge *F^a) \otimes e_a^1 \vee e_b^2\Big] \otimes E_a \vee E_b.$$

2.4.3 Elaborating the field equations

If we want to consider a set of independent solutions, then in the above expression we take the trace tr over the indeces of $E_a \vee E_b$. The compact writing of this condition reads

$$tr(\vee,\vee)(\delta\Omega, *\Omega) = 0, \tag{2.40}$$

which is equivalent to the equations

$$\delta F^a \wedge *F^a = 0,\ \delta * F^a \wedge ** F^a = 0,\ \delta F^a \wedge ** F^a + \delta * F^a \wedge *F^a = 0. \tag{2.41}$$

Clearly, in this case the full energy-momentum tensor Q_μ^ν will be a sum of all energy tensors $(Q^a)_\mu^\nu$ of the single solutions.

The general equations are written down as follows:

$$(\vee,\vee)(\delta\Omega, *\Omega) = 0. \tag{2.42}$$

The equivalent (component-wise) form of (2.42) reads

$$\delta F^a \wedge *F^a = 0,\ \delta * F^a \wedge ** F^a = 0,\ \delta F^a \wedge ** F^a + \delta * F^a \wedge *F^a = 0,$$

$$\delta F^a \wedge *F^b + \delta F^b \wedge *F^a = 0,\ \delta * F^a \wedge ** F^b + \delta * F^b \wedge ** F^a = 0,$$
$$\delta F^a \wedge ** F^b + \delta * F^b \wedge ** F^a + \delta * F^a \wedge *F^b + \delta * F^b \wedge *F^a = 0.$$

Let now F^a, $a = 1, 2, ..., N$ define a solution of the above system of equations (2.42). We are going to show that the linear combination with constant coefficients λ_a

$$F = \sum_{a=1}^{N} \lambda_a F^a$$

satisfies the equations: $\delta F \wedge *F = 0,\ \delta * F \wedge ** F = 0,$
$\delta F \wedge ** F + \delta * F \wedge *F = 0$. In fact

$$\delta F \wedge *F = \sum_{a=1}^{N} (\lambda_a)^2 (\delta F^a \wedge *F^a) + \sum_{a<b=1}^{N} \lambda_a \lambda_b (\delta F^a \wedge *F^b + \delta F^b \wedge *F^a),$$

$$\delta * F \wedge ** F = \sum_{a=1}^{N} (\lambda_a)^2 (\delta * F^a \wedge ** F^a) + \sum_{a<b=1}^{N} \lambda_a \lambda_b (\delta * F^a \wedge ** F^b + \delta * F^b \wedge ** F^a),$$

$$\delta F \wedge ** F + \delta * F \wedge *F = \sum_{a=1}^{N} (\lambda_a)^2 (\delta F^a \wedge ** F^a) +$$

$$+\sum_{a<b=1}^{N}\lambda_a\lambda_b(\delta F^a\wedge **F^b+\delta F^b\wedge **F^a)+$$

$$+\sum_{a=1}^{N}(\lambda_a)^2(\delta * F^a\wedge *F^a)+\sum_{a<b=1}^{N}\lambda_a\lambda_b(\delta * F^a\wedge *F^b+\delta * F^b\wedge *F^a)=$$

$$=\sum_{a=1}^{N}(\lambda_a)^2(\delta F^a\wedge **F^a+\delta * F^a\wedge *F^a)+$$

$$+\sum_{a<b=1}^{N}\lambda_a\lambda_b(\delta F^a\wedge **F^b+\delta F^b\wedge **F^a+\delta * F^a\wedge *F^b+\delta * F^b\wedge *F^a).$$

Obviously, the component-wise writing down of the equations (2.42) shows that every addend is equal to zero. This result can be interpreted as some particular "superposition principle", i.e. if we have finite number of solutions F^a of the system

$$\delta F\wedge *F=0,\ \delta * F\wedge **F=0,\ \delta F\wedge **F+\delta * F\wedge *F=0, \tag{2.43}$$

which solutions satisfy additionally the equations

$$\delta F^a\wedge *F^b+\delta F^b\wedge *F^a=0,\ \delta * F^a\wedge **F^b+\delta * F^b\wedge **F^a=0, \tag{2.44}$$

$$\delta F^a\wedge **F^b+\delta * F^b\wedge **F^a+\delta * F^a\wedge *F^b+\delta * F^b\wedge *F^a=0, \tag{2.45}$$

then the 2-form $F=\sum_{a=1}^{N}\lambda_a F^a$ is again a solution of (2.43). Then, clearly, if F and G are 2 solutions of (2.43) and satisfy (2.44) and (2.45), the new solution $(F+G)$ of (2.43) is naturally endowed with the following energy-momentum tensor

$$Q_{\mu\nu}=\frac{1}{4\pi}\Big[-(F+G)_{\mu\sigma}(F+G)_\nu^\sigma\Big].$$

In the general case we'll have

$$Q_{\mu\nu}=\frac{1}{4\pi}\left[-\left(\sum_{a=1}^{N}\lambda_a F^a\right)_{\mu\sigma}\left(\sum_{a=1}^{N}\lambda_a F^a\right)_\nu^\sigma\right]$$

In this way we can compute the corresponding "interference terms". In particular, the "interference" energy density is obtained proportional to $-2F_{4\sigma}G^{4\sigma}$.

2.4.4 Coherence and interference

We consider now two photon-like solutions F_1 and F_2 of equations (2.43), propagating along the same direction. We choose this direction for the z-axis of our coordinate system. We are going to find what additional conditions on these solutions come from the additional equations (2.44) and (2.45). We assume also, that the 3-regions, where the two amplitudes ϕ_1 and ϕ_2 are different from zero have non-empty intersection, because otherwise, the interference term is equal to zero.

$$F_1 = \varepsilon_1 u_1 dx\wedge dz + u_1 dx\wedge d\xi + \varepsilon_1 p_1 dy\wedge dz + p_1 dy\wedge d\xi$$

$$F_2 = \varepsilon_2 u_2 dx\wedge dz + u_2 dx\wedge d\xi + \varepsilon_2 p_2 dy\wedge dz + p_2 dy\wedge d\xi,$$

where

$$u_1 = \phi_1 cos\left(\frac{\kappa_1\nu_1}{c}\xi + b_1\right),\ p_1 = \phi_1 sin\left(\frac{\kappa_1\nu_1}{c}\xi + b_1\right),$$

$$u_2 = \phi_2 cos\left(\frac{\kappa_2\nu_2}{c}\xi + b_2\right),\ p_2 = \phi_2 sin\left(\frac{\kappa_2\nu_2}{c}\xi + b_2\right).$$

After an elementary computation we obtain

$$\delta F_1\wedge *F_2 + \delta F_2\wedge *F_1 = \left(\frac{\kappa_1\nu_1}{c} - \frac{\kappa_2\nu_2}{c}\right)(u_1p_2 - u_2p_1)dx\wedge dy\wedge dz +$$

$$+\left(\varepsilon_1\frac{\kappa_1\nu_1}{c} - \varepsilon_2\frac{\kappa_2\nu_2}{c}\right)(u_1p_2 - u_2p_1)dx\wedge dy\wedge d\xi +$$

$$+\left[p_1\left(u_{1x} + p_{1y}\right) + p_2\left(u_{2x} + p_{2y}\right)\right](\varepsilon_1\varepsilon_2 - 1)dx\wedge dz\wedge d\xi +$$

$$+\left[u_1\left(u_{1x} + p_{1y}\right) + u_2\left(u_{2x} + p_{2y}\right)\right](1 - \varepsilon_1\varepsilon_2)dy\wedge dz\wedge d\xi.$$

Since

$$u_1p_2 - u_2p_1 = \phi_1\phi_2 sin\left[\left(\frac{\kappa_2\nu_2}{c} - \frac{\kappa_1\nu_1}{c}\right)\xi + b_2 - b_1\right] \neq 0,$$

the coefficient before $dx\wedge dy\wedge dz$ will be equal to zero only if $\kappa_1\nu_1 = \kappa_2\nu_2$. But ν_1 and ν_2 are positive quantities, so it is necessary $\kappa_1 = \kappa_2,\ \nu_1 = \nu_2$. Now, the coefficient in front of $dx\wedge dy\wedge d\xi$ will become zero if $\varepsilon_1 = \varepsilon_2$. From this last relation it follows that the other two coefficients, obviously, are also zero. A corresponding computation shows that the so obtained conditions

$$\nu_1 = \nu_2,\ \varepsilon_1 = \varepsilon_2,\ \kappa_1 = \kappa_2 \tag{2.46}$$

are sufficient for F_1 and F_2 to satisfy the rest two equations of (2.44) and (2.45). Hence, *if the 2-form*

$$\Omega = (F_1 \otimes e_1 + *F_1 \otimes e_2) \otimes E_1 + (F_2 \otimes k_1 + *F_2 \otimes k_2) \otimes E_2$$

satisfies equations (2.42), *then the 2-form* $F = F_1 + F_2$ *is a solution of our initial equations*

$$\delta F \wedge *F = 0, \ \delta * F \wedge ** F = 0, \ \delta F \wedge ** F + \delta * F \wedge *F = 0,$$

but not of photon-like type, the coherence conditions (2.42) *are satisfied and the interference of the two fields* F_1 and F_2 *is possible.* As for the "interference" energy density we obtain the well known from CED expression

$$W_{12} = \phi_1^2 + \phi_2^2 + 2\phi_1\phi_2 cos(b_2 - b_1)$$

from which the classical interference picture is readily obtained.

Chapter 3

Extended Electrodynamics in Media

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3.1 Basic equations

3.1.1 Preliminary remarks

Recall from subsec.(1.5.3) that when we talk about a *medium* in EED, we mean any continuous, i.e. spatially distributed, physical object, exchanging energy-momentum with the available in the same region EM-field Ω. Formally, the medium is described by some mathematical object and, when this object is chosen, we talk about *external* or *outer* field. When interaction between Ω and the outer field takes place, of basic importance for the theory is how Ω and the external field participate in the expression, defining the exchanged energy-momentum in an unit 4-volume. According to the hypotheses we made in subsec.(1.5.3) the EM-field Ω participates directly in this expression, while the exteriour field participates in this expression through specially constructed two $\mathcal{R}^2$-valued 1-forms, and these 1-forms may depend on the derivatives of the external field too. The vector-components of these $\mathcal{R}^2$-valued 1-forms were called *currents*, since the classical *current*, considered as a 1-form, may be considered as a particular case. Taking into account the mathematical model of the EM-field, as well as the experience with Maxwell's theory, we postulated the expression (1.42)

$$\vee(\Phi, *\pi_1\Omega) \ + \ \vee(\Psi, *\pi_2\Omega)$$

as sufficiently general and adequate to describe large enough class of exchange processes, i.e. interaction of Ω with outer fields.

Such an approach, when the algebraic character and the differential equations for the exteriour field are not known, needs a new, general enough and sufficiently adaptable viewpoint, expressing a definite comprehension for the character of the physical processes considered, as well as general enough and adequate enough mathematical facts. Such an adequacy must give reasons for definite hypotheses, and, finally, to result in writing down definite equations for the currents, no matter what the particular nature of the currents is. We note, that we do not require and do not forbid the four currents α^i to have zero divergence. In our opinion, the new facts to be used as fully as possible, are that their number is *more than one* , that every current realizes a separate energy exchange channel, and that there should be some correlations among them. It is naturally to expect such correlations among the four currents to exist since the exchange processes occur locally and on the other side of this exchange stays just one physical object - the EM-field Ω. These correlations must be organized in such a way, that to incorporate the special case of only one current different from zero, as it is in CED. Of course, we have to remember that in CED the current is a *vector field*, while our currents are 1-forms. These are different objects although the available isomorphism through the pseudometric η, and this difference will be explicitly taken into account in our approach.

3.1.2 Maxwell's theory as a particular case

Let's recall some important features of CED. First, *the EM-field exchanges energy-momentum only through F and does not exchange energy-momentum through $*F$*. Formally, this is accounted by the equation $\delta * F = 0$, which contains (in differential form) the Faraday's induction law. In other words, CED assumes, that this experimentally established fact for *some* media, holds for *all* media. As we know, this assumption legalizes the $U(1)$-gauge interpretation of CED, where the equation $\delta * F = 0$ acquires the geometrical interpretation of Bianchi's identity for the abelean group $U(1)$.

Second, an energy-momentum exchange may be realized *only with free or bound electric charges.* The formal description of this exchange was explained and commented in subsec. (1.1.3). The second equation of CED,

$$\delta F = 4\pi(j + j_b),$$

identifies the pure field quantity δF with the outer quantity *current*, which characterizes the distribution and mechanical behaviour of the charge carriers. To what extent such an identification is admissible is a personal view, and we are not going to comment it. According to us much more natural is to write down a relationship having the sense of local energy-momentum balance. In other words, the same (exchanged) quantity of energy-momentum to be written down in two ways: in the first way, by means of the components $F_{\mu\nu}$ and their derivatives only, and in the second way, through expressions where the outer field components necessarily take part. Then, according to the local conservation law, the two expressions are equalized. Namely such an approach we have realized in EED.

The third feature we recall is the lack of some general enough and common approach for determination of the full current $(j_{free} + j_{bound})$. As we mentioned earlier, the series developments of the polarization vector P and magnetization vector $\mathcal{M}$ (or the 2-form S) with respect to E and B and their derivatives may be a felicitous working skill, but it is not a perspective theoretical idea. For example, this approach is not applicable for strongly nonhomogeneous media, while a local energy-momentum balance equation can be written down always and to be particularized and made more precise in the course of work.

We see now how, from formal point of view, CED is incorporated in EED. The mathematical expression for the exchanged energy-momentum in CED is

$$4\pi F_{\mu\nu}(\delta S + J)^\nu dx^\mu.$$

In our approach, using the defined by the field Ω basis (e_1, e_2), this expression is represented consecutively as follows:

$$4\pi F_{\mu\nu}(\delta S + J)^\nu dx^\mu \otimes e_1 \vee e_1 = *[4\pi(\delta S + J) \wedge *F] \otimes e_1 \vee e_1 =$$

$$= * \vee [4\pi(\delta S + J) \otimes e_1, *F \otimes e_1] = * \vee [\pi_1\Phi, \pi_1 * \Omega],$$

where $\pi_1\Phi = 4\pi(\delta S + J) \otimes e_1$. In this way, using the notations of EED, we get

$$\vee(\delta\Omega, *\Omega) = \vee(\pi_1\Phi, \pi_1 * \Omega) + \vee(\pi_1\Psi, \pi_2 * \Omega), \quad \pi_1\Phi = \pi_1\Psi.$$

We see that there are no terms, describing an exchange through $*F$, and the redistribution energy-momentum equation reduces to one of the real exchange equations. So, CED reflects the following additional requirements to the equations of EED: $\Phi = \alpha^1 \otimes e_1,\ \Psi = \alpha^3 \otimes e_1,\ \alpha^1 = \alpha^3 = 4\pi(\delta S + J),\ \alpha^2 =$

$\alpha^4 = 0$. The very Maxwell's equations $\delta * F = 0,\ \delta F = 4\pi(\delta S + J)$ may be written also as: $\pi_2 \delta\Omega = 0,\ \pi_1 \delta\Omega = 4\pi(\delta S + J) \otimes e_1$.

3.1.3 Component form of the equations

The coordinate free written relationship (1.43)

$$\vee(\delta\Omega, *\Omega) = \vee(\Phi, *\pi_1\Omega) \ + \ \vee(\Psi, *\pi_2\Omega)$$

is equivalent to the following relations:

$$\begin{gathered} \delta F \wedge *F = \alpha^1 \wedge *F,\ \delta * F \wedge * * F = \alpha^4 \wedge * * F, \\ \delta F \wedge * * F + \delta * F \wedge *F = \alpha^3 \wedge * * F + \alpha^2 \wedge *F, \end{gathered} \tag{3.1}$$

or, in components

$$\begin{gathered} F_{\mu\nu}(\delta F)^\nu = F_{\mu\nu}(\alpha^1)^\nu,\ (*F)_{\mu\nu}(\delta * F)^\nu = (*F)_{\mu\nu}(\alpha^4)^\nu, \\ F_{\mu\nu}(\delta * F)^\nu + (*F)_{\mu\nu}(\delta F)^\nu = (*F)_{\mu\nu}(\alpha^3)^\nu + F_{\mu\nu}(\alpha^2)^\nu. \end{gathered} \tag{3.2}$$

Moving everything on the left, we get

$$(\delta F - \alpha^1) \wedge *F = 0,\ (\delta * F - \alpha^4) \wedge * * F = 0,$$

$$(\delta F - \alpha^3) \wedge * * F + (\delta * F - \alpha^2) \wedge *F = 0,$$

or in components

$$F_{\mu\nu}(\delta F - \alpha^1)^\nu = 0,\ (*F)_{\mu\nu}(\delta * F - \alpha^4)^\nu = 0,$$

$$F_{\mu\nu}(\delta * F - \alpha^2)^\nu + (*F)_{\mu\nu}(\delta F - \alpha^3)^\nu = 0.$$

Summing up the two equations

$$F_{\mu\nu}(\delta F)^\nu = F_{\mu\nu}(\alpha^1)^\nu,\ (*F)_{\mu\nu}(\delta * F)^\nu = (*F)_{\mu\nu}(\alpha^4)^\nu$$

we obtain

$$F_{\mu\nu}(\delta F)^\nu + (*F)_{\mu\nu}(\delta * F)^\nu = \nabla_\nu Q_\mu^\nu = F_{\mu\nu}(\alpha^1)^\nu + (*F)_{\mu\nu}(\alpha^4)^\nu.$$

This relation shows that the sum

$$F_{\mu\nu}(\alpha^1)^\nu + (*F)_{\mu\nu}(\alpha^4)^\nu$$

is a divergence of a 2-tensor, which we denote by $-P_\mu^\nu$. In this way we obtain the local conservation law

$$\nabla_\nu(Q_\mu^\nu + P_\mu^\nu) = 0. \tag{3.3}$$

Thus, we get the possibility to introduce the full energy-momentum tensor

$$T_\mu^\nu = Q_\mu^\nu + P_\mu^\nu,$$

where P_μ^ν is interpreted as *interaction energy-momentum tensor*. Clearly, P_μ^ν can not be determined uniquely in this way.

So, according to (3.2), for the 22 functions $F_{\mu\nu}, (\alpha^i)_\mu$ we have 12 equations, and these 12 equations are differential with respect to $F_{\mu\nu}$ and algebraic with respect to $(\alpha^i)_\mu$. Our purpose now is to try to write down differential equations for the components of the 4 currents. The leading idea in pursuing this goal will be to establish a correspondence between the physical concept of *non-dissipation* and the mathematical concept of *integrability of Pfaff systems*. The suggestion to look for such a correspondence comes from the following considerations.

Recall from the theory of the ordinary differential equations (or vector fields), that every solution of a system of ordinary differential equations (ODE) defines a local (with respect to the parameter on the trajectory) group of transformations, frequently called *local flow*. This means, in particular, that the motion along the trajectory is admissible in the two directions: we have a *reversible* phenomenon, which has the physical interpretation of *lack of losses* (energy-momentum losses are meant). Assuming this system of ODE describes *fully* the process of motion of a small piece of matter (particle), we assume at the same time, that *all* energy-momentum exchanges between the particle and the outer field are taken into account, i.e. we have assumed that *there is no dissipation*. In other words, *the physical assumption for the lack of dissipation is mathematically expressed by the existence of solution - local flow, having definite group properties.* The existence of such a local flow is guaranteed by the corresponding theorem for existence and uniqueness of a solution at given initial conditions. This correspondence between the mathematical fact *integrability* and the physical fact *lack of dissipation* in the simple case "motion of a particle" , we want to generalize in an appropriate way, having in view possible applications in more complicated physical systems, in particular, the physical situation we are going to describe: interaction of the field Ω with some outer field, represented in the exchange process by the

four 1-forms α^i. This will allow to write down equations for α^i in a direct way. Of course, in the real world there is always dissipation, and following this idea we are going to take into account its neglecting as conditions (i.e. equations) on the currents. As it is well known, the mathematicians have made serious steps towards studding and formulation of criteria for integrability of partial differential equations, so it looks unreasonable to close eyes before the available and represented in an appropriate form mathematical results.

On the other hand it is interesting, and may be suggesting, to note the following . In physics we have two *universal* things: *dissipation* and *gravitation*. We are going now to establish a correspondence between the physical notion of *dissipation* and the mathematical concept of *non-integrability*. As we know, the mathematical non-integrability is measured by the concept of *curvature*. General theory of Relativity describes gravitation by means of Riemannian curvature. The circle will be closed if we connect the universal property of any real physical process to dissipate energy-momentum with the only known so far universal interaction in nature, the gravitation.

3.2 The Frobenius Integrability and Dissipation

3.2.1 Integrable distributions and curvature

The problem for integration of a system of partial differential equations of the kind

$$\frac{\partial y^a}{\partial x^i} = f_i^a(x^k, y^b), \;\; i,k = 1,...,p; \;\; a,b = 1,...,q, \tag{3.4}$$

where $f_i^a(x^k, y^b)$ are given functions, obeying some definite smoothness conditions, has brought about to the formulation of a number of concepts, which in turn have become generators of ideas and directions, as well as have shown an wide applicability in many branches of modern mathematics. A particular case of the above system (nonlinear in general) of equations is when there is only one independent variable, i.e. when all x^i are reduced to x^1, which is usually denoted by t and the system acquires the form

$$\frac{dy^a}{dt} = f^a(y^b, t), \;\; a,b = 1,...,q. \tag{3.5}$$

We are going to give now the system of concepts used in considering the integrability problems for the equations (3.4) and (3.5), making use of the geometric language of manifolds theory.

Let X be a vector field on the q-dimensional manifold M and the map $c : I \to M$, where I is an open interval in $\mathcal{R}$, defines a smooth curve in M. Then if X^a are the components of X with respect to the local coordinates $(y^1, ..., y^q)$ and the equality $c'(t) = V(c(t))$ holds for every $t \in I$, or in local coordinates,

$$\frac{dy^a}{dt} = X^a(y^b),$$

$c(t)$ is called *integral curve* of the vector field X. As it is seen, the difference with (3.5) is in the additional dependence of the right side of (3.5) on the independent variable t. Mathematics approaches these situations in an unified way as follows. The product $\mathcal{R} \times M$ is considered and the important theorem for uniqueness and existence of a solution is proved: For every point $p \in M$ and point $\tau \in \mathcal{R}$ there exist a vicinity U of p, a positive number ε and a smooth map $\Phi : (\tau - \varepsilon, \tau + \varepsilon) \times U \to M$, $\Phi : (t, y) \to \varphi_t(y)$, such that for every point $y \in U$ the following conditions are met: $\varphi_\tau(y) = y,\ t \to \varphi_t(y)$ is an integral curve of X, passing through the point $y \in M$; besides, if two such integral curves of X have at least one common point, they coincide. Moreover, if $(t', y),\ (t+t', y)$ and $(t, \varphi(y))$ are points of a vicinity U' of $\{0\} \times \mathcal{R}$ in $\mathcal{R} \times M$, we have $\varphi_{t+t'}(y) = \varphi_t(\varphi_{t'}(y))$. This last relation gives the local group action: for every $t \in I$ we have the local diffeomorphism $\varphi_t : U \to \varphi_t(U)$. So, through every point of M there passes only one trajectory of X and in this way the manifold M is foliated to non-crossing trajectories - 1-dimensional manifolds, and these 1-dimensional manifolds define all trajectories of the defined by the vector field X system of ODE. This fibering of M to non-intersecting submanifolds, the union of which gives the whole manifold M, together with uniting t and $y(t)$ in one manifold, is the leading idea in treating the system of partial differential equations (3.4), where the number of the independent variables is more than 1, but finite. For example, if we consider two vector fields on M, then through every point of M two trajectories will pass and the question: when a 2-dimensional surface, passing through a given point can be built, and such that the representatives of the two vector fields at every point of this 2-surface to be tangent to the surface, naturally arises. Mathematics sets this question for p-dimensional surfaces, builds the necessary concepts and proves the corresponding theorems. These theorems formulate the criteria for integrability of (3.4), and are known in the litera-

ture as *Frobenius theorems*. For simplicity, further we consider regions of the space $\mathcal{R}^p \times \mathcal{R}^q$, but this is not essentially important since the Frobenius theorems are local statements, so the results will hold for any $(p+q)$-dimensional manifold.

Let U be a region in $\mathcal{R}^p \times \mathcal{R}^q$, and $(x^1, ..., x^p, y^1 = x^{p+1}, ..., y^q = x^{p+q})$ are the canonical coordinates. We set the question: for which points (x_0, y_0) of U the system of equations (3.4) has a solution $y^a = \varphi^a(x^i)$, defined for points x, sufficiently close to x_0 and satisfying the initial condition $\varphi(x_0) = y_0$? The answer to this question is: for this it is necessary and sufficient the functions f_i^a on the right hand side of (3.4) to satisfy the following conditions:

$$\frac{\partial f_i^a}{\partial x^j}(x,y) + \frac{\partial f_i^a}{\partial y^b}(x,y).f_j^b(x,y) = \frac{\partial f_j^a}{\partial x^i}(x,y) + \frac{\partial f_j^a}{\partial y^b}(x,y).f_i^b(x,y). \qquad (3.6)$$

This relation is obtained as a consequence of two basic steps: first, equalizing the mixed partial derivatives of y^a with respect to x^i and x^j, second, replacing the obtained first derivatives of y^a with respect to x^i on the right hand side of (3.4) again from the system (3.4). This second step means, that everywhere in (3.6) y are considered as functions of x, i.e. there is no explicit dependence on y. If the functions f_i^a satisfy the equations (3.6), the system (3.4) is called *completely integrable*. In order to give a coordinate free formulation of (3.6) and to introduce the *curvature, as a measure for non-integrability* of (3.4), we shall first sketch the necessary terminology.

Let M be an arbitrary $n = p + q$ dimensional manifold. At every point $x \in M$ the tangent space $T_x(M)$ is defined. The union of all these spaces with respect to the points of M defines the *tangent bundle*. On the other hand, the union of the co-tangent spaces $T_x^*(M)$ defines the *co-tangent bundle*. At every point now of M we separate a p dimensional subspace $\Delta_x(M)$ of $T_x(M)$ in a smooth way, i.e. the map $x \rightarrow \Delta_x$ ix smooth. If this is done we say that a p-dimensional *distribution* Δ on M is defined. From the elementary linear algebra we know that every p-dimensional subspace Δ_x of $T_x(M)$ defines unique $(n-p) = q$ dimensional subspace Δ_x^* of the dual to $T_x(M)$ space $T_x^*(M)$, such that all elements of Δ_x^* annihilate (i.e. send to zero) all elements of Δ_x. In this way we get a q-dimensional *co-distribution* Δ^* on M. We consider those vector fields, the representatives of which at every point are elements of the distribution Δ, and those 1-forms, the representatives of which at every point are elements of the co-distribution Δ^*. Clearly, every system of p independent vector fields, belonging to Δ, defines Δ equally well, and in this case we call such a system a *differential*

p-system $\mathcal{P}$ on M. The corresponding system $\mathcal{P}^*$ of q independent 1-forms is called q-dimensional *Pfaff system*. Clearly, if $\alpha \in \mathcal{P}^*$ and $X \in \mathcal{P}$, then $\alpha(X) = 0$.

Similarly to the integral curves of vector fields, the concept of *integral manifold* of a p-dimensional differential system is introduced. Namely, a p-dimensional submanifold V^p of M is called *integral manifold* for the p-dimensional differential system $\mathcal{P}$, or for the p-dimensional distribution Δ,to which $\mathcal{P}$ belongs, if the tangent spaces of V^p at every point coincide with the subspaces of the distribution Δ at this point. In this case V^p is called also integral manifold for the q-dimensional Pfaff system $\mathcal{P}^*$. If through every point of M there passes an integral manifold for $\mathcal{P}$, then $\mathcal{P}$ and $\mathcal{P}^*$ are called *completely integrable.*

Now we shall formulate the Frobenius theorems for integrability.

A differential system $\mathcal{P}$ is completely integrable if and only if the Lie bracket of any two vector fields, belonging to $\mathcal{P}$, also belongs to $\mathcal{P}$.

So, if $(X_1, ..., X_p)$ generate the completely integrable differential system $\mathcal{P}$, then

$$[X_i, X_j] = C^k_{ij} X_k, \tag{3.7}$$

where the coefficients C^k_{ij} depend on the point.

This criterion is not quite convenient to use because its usage presupposes the knowledge of the functions C^k_{ij}. It turns out that the corresponding criterion for Pfaff systems does not require any additional information. In fact, let the 1-forms $(\alpha^1, ..., \alpha^q)$ define the q-dimensional Pfaff system $\mathcal{P}^*$. The following criterion holds (the dual Frobenius theorem):

The Pfaff system $\mathcal{P}^*$ is completely integrable if and only if

$$\mathbf{d}\alpha^a = K^a_{bc} \alpha^b \wedge \alpha^c, \ b < c. \tag{3.8}$$

It is easily shown, that the above equations are equivalent to the following equations:

$$(\mathbf{d}\alpha^a) \wedge \alpha^1 \wedge \ \ldots \wedge \alpha^a \wedge \ \ldots \wedge \alpha^q = 0, \ a = 1, ..., q, \tag{3.9}$$

which, obviously, do not depend on any coefficients.

When a given Pfaff system $\mathcal{P}^*$, or the corresponding differential system $\mathcal{P}$, are not integrable, then the relations (3.7)-(3.9) *are not fulfilled*. From formal point of view this means that there is at least one couple of vector fields (X, Y), belonging to $\mathcal{P}$, such that the Lie bracket $[X, Y]$ *does not belong* to $\mathcal{P}$. Therefore, if at the corresponding point $x \in M$ we choose a basis of $T_x(M)$ such, that the first p basis vectors to form a basis of Δ_x, then $[X, Y]$ will have nonzero components with respect to those basis vectors, which belong to some complimentary to Δ_x subspace $\Gamma_x : \ T_x(M) = \Delta_x \oplus \Gamma_x$. If the spaces $\Gamma_x, x \in M$ are beforehand given we may consider the *projection* of the Lie bracket $[X, Y]$ onto these complimentary subspaces. Then this projection is defined uniquely by the choice of the distribution Δ, so it is a natural measure for the non-integrability of Δ. If M has the structure of *bundle space*, which means that a base-space B, $dimB = p$, is given, and a smooth map $\pi : \ M \rightarrow B$ of maximal rank, i.e. $rank(d\pi) = p < n$, is given, then the subspaces $V_x = Ker(d\pi)_x \subset T_x(M)$ are naturally separated. It is easily shown that these subspaces, called usually *vertical*, form an integrable distribution. If we orient our interest towards distributions $\Delta(M)$, which are complimentary to vertical distributions and are usually called *horizontal*, then the non-integrability of $\Delta(M)$ will be determined entirely by the vertical projection $v : T(M) \rightarrow V(M)$, defined by the definition of $\Delta(M)$ and considered on the various Lie brackets of horizontal vector fields. It is clear now, that the *curvature* $\mathcal{K}$ of the horizontal distribution Δ is defined by

$$\mathcal{K}(X, Y) = v([X, Y]), \ X, Y \in \Delta. \tag{3.10}$$

It is also clear, that the curvature $\mathcal{K}$ is a 2-form on M with values in the tangent bundle $T(M)$, and $\mathcal{K}$ is reduced to identity on Lie brackets of vertical vectors. In fact, if f is a smooth function and X, Y are two horizontal vector fields, then

$$\mathcal{K}(X, fY) = v([X, fY]) = v(f[X, Y] + X(f)Y) =$$

$$= fv([X, Y]) + X(f)v(Y) = f\mathcal{K}(X, Y)$$

because $v(Y) = 0, \ Y - horizontal$.

The corresponding q-dimensional co-distribution Δ^* (or Pfaff system $\mathcal{P}^*$) is defined locally by the 1-forms $(\theta^1, ..., \theta^q)$, such that $\theta^a(X) = 0$ for all horizontal vector fields X. In this case the 1-forms θ^a are called *vertical*, and clearly, they depend on the choice of the horizontal distribution Δ. In these

terms the non-integrability of Δ means

$$\mathbf{d}\theta^a \neq K^a_{bc}\theta^b \wedge \theta^c,\ b<c.$$

This non-equality means that at least one of the 2-forms $\mathbf{d}\theta^a$ is not vertical, i.e. it has a nonzero horizontal projection $H^*\mathbf{d}\theta^a$, which means that it does not annihilate all horizontal vectors. In these terms it is naturally to define the curvature by

$$H^*\mathbf{d}\theta^a = \mathbf{d}\theta^a - K^a_{bc}\theta^b \wedge \theta^c,\ b<c.$$

Further we shall see how this picture is defined by the equations (3.4).

Let's begin with the remark that the consideration of bundle spaces only, is not a limitation and does not bounds the results, because we want through every point of M to pass locally only one integral manifold of a given horizontal distribution and this integral manifold will be diffeomorphic to an open set in the base manifold B. And this is just what is guaranteed by the bundle structure of M: every point $b \in B$ has an open vicinity U, and $\pi^{-1}(U)$ is diffeomorphic to the direct product $U \times N$, where N is a q-dimensional manifold, called *standard fiber*. This bundle structure allows canonical (or bundle adapted) local coordinates (x^i, y^a) to be introduced, reflecting the local-product nature of M: $x^i = z^i \circ \pi$, where z^i are local coordinates on $U \subset B$, and y^a are local coordinates on N. In these coordinates a local basis of the vertical vector fields is given by

$$\frac{\partial}{\partial y^a},\ a = p+1, ..., p+q = n.$$

Now, making use the equations of the system (3.4), i.e. the functions f^a_i, we have to define the horizontal spaces at every point of $\pi^{-1}(U) \subset M$, i.e. local linearly independent vector fields $X_i, i = 1, ..., p$. The definition is

$$X_i = \frac{\partial}{\partial x^i} + f^a_i \frac{\partial}{\partial y^a}. \tag{3.11}$$

The corresponding Pfaff system shall consist of 1-forms $\theta^a, a = p+1, ..., p+q$ and is defined by

$$\theta^a = dy^a - f^a_i dx^i. \tag{3.12}$$

In fact,

$$\theta^a(X_i) = dy^a\left(\frac{\partial}{\partial x^i}\right) + dy^a\left(f^b_i \frac{\partial}{\partial y^b}\right) - f^a_j dx^j\left(\frac{\partial}{\partial x^i}\right) - f^a_j dx^j\left(f^b_i \frac{\partial}{\partial y^b}\right) =$$

$$= 0 + f_i^b \delta_b^a - f_j^a \delta_i^j - 0 = 0.$$

Remark. The coordinate 1-forms dy^a are not vertical with respect to the so defined horizontal distribution.

In this way in the bundle-adapted coordinate systems the system (3.4) defines unique horizontal distribution. On the other hand, if a horizontal distribution is given and the corresponding vertical Pfaff system admits at least i basis, then this basis may be chosen of the kind (3.12) always. Moreover, in this kind it is unique. In fact, let $(\theta'^1, ..., \theta'^q)$ be any local basis of Δ^*. Then in the adapted coordinates we'll have

$$\theta'^a = A_b^a dy^b + B_i^a dx^i,$$

where A_b^a, B_i^a are functions on M. We shall show that the matrix A_b^a has non-zero determinant, i.e. it is non-degenerate. Assuming the opposite, we could find scalars λ_a, not all of which are equal to zero, and such that the equality $\lambda_a A_b^a = 0$ holds. We multiply now the above equality by λ^a and sum up with respect to a. We get

$$\lambda_a \theta'^a = \lambda_a B_i^a dx^i.$$

Note now that on the right hand side of this last relation we have a horizontal 1-form, while on the left hand side we have a vertical 1-form. This is impossible by construction, so our assumption is not true, i.e. the inverse matrix $(A_b^a)^{-1}$ exists, so multiplying on the left θ'^a by $(A_b^a)^{-1}$ and putting $(A_b^a)^{-1}\theta'^b = \theta^a$ we obtain

$$\theta^a = dy^a + (A_b^a)^{-1} B_i^b dx^i.$$

We denote now $(A_b^a)^{-1} B_i^b = -f_i^a$ and get what we need. The uniqueness part of the assertion is proved as follows. Assume there is another basis $(\theta^a)''$ of the same kind. So, there must be a non-degenerate matrix C_b^a, such that $(\theta^a)'' = C_b^a \theta^b$. We get

$$dy^a - (f_i^a)'' dx^i = C_b^a dy^b - C_b^a f_i^b dx^i,$$

and from this relation it follows that the matrix C_b^a is the unit one.

Let's see now the explicit relation between the integrability condition (3.6) of the system (3.4) and the curvature $H^*\mathbf{d}\theta^a$ of the defined by this system horizontal distribution. We obtain

$$\mathbf{d}\theta^a = -\frac{\partial f_i^a}{\partial x^j} dx^j \wedge dx^i - \frac{\partial f_i^a}{\partial y^b} dy^b \wedge dx^i.$$

In order to define the horizontal projection of $\mathbf{d}\theta^a$ it is necessary to separate the horizontal part of dy^a. Since θ^a are vertical, from their explicit form is seen that the horizontal part of dy^a is just $f_i^a dx^i$. That's why for the curvature $H^*\mathbf{d}\theta^a$ we get

$$H^*\mathbf{d}\theta^a = \left(\frac{\partial f_i^a}{\partial x^j} - \frac{\partial f_j^a}{\partial x^i} + \frac{\partial f_i^a}{\partial y^b} f_j^b - \frac{\partial f_j^a}{\partial y^b} f_i^b\right) dx^i \wedge dx^j, \; i < j. \tag{3.13}$$

It is clearly seen that the integrability condition (3.6) coincides with the requirement for zero curvature. The replacing of dy^a, making use of the system (3.4), in order to obtain (3.6), acquires now the status of "horizontal projection".

We verify now that the curvature, defined by $v([X_i, X_j])$ gives the same result.

$$v([X_i, X_j]) = v\left(\left[\frac{\partial}{\partial x^i} + f_i^a \frac{\partial}{\partial y^a}, \frac{\partial}{\partial x^j} + f_j^b \frac{\partial}{\partial y^b}\right]\right) =$$

$$= v\left(\frac{\partial f_j^b}{\partial x^i}\frac{\partial}{\partial y^b} - \frac{\partial f_i^a}{\partial x^j}\frac{\partial}{\partial y^a} + f_i^a \frac{\partial f_j^b}{\partial y^a}\frac{\partial}{\partial y^b} - f_i^b \frac{\partial f_i^a}{\partial y^b}\frac{\partial}{\partial y^a}\right) =$$

$$= \left(\frac{\partial f_j^a}{\partial x^i} - \frac{\partial f_i^a}{\partial x^j} + \frac{\partial f_j^a}{\partial y^b} f_i^b - \frac{\partial f_i^a}{\partial y^b} f_j^b\right)\frac{\partial}{\partial y^a}.$$

The interchange of the indices i and j does not impact the equivalence of this result to the above obtained for $H^*\mathbf{d}\theta^a$.

3.2.2 Physical interpretation

As it was noticed in the preceding subsection we want to connect the physical concept of *dissipation* with the mathematical concept of *Frobenius non-integrability of Pfaff systems*. The availability of a well defined mathematical quantity as the *curvature*, which has been extensively used from the beginning of this century in mathematics (differential geometry and differential topology) and theoretical physics (General Relativity (GR) and Gauge theories) makes the things more attractive in view of its wider use in physics. In General Relativity curvatures of Riemannean connections are used, and because of the stress on the metric tensor as a potential of the curvature, there has not been paid enough attention to the original meaning of the curvature, namely as a *measure for non-integrability*. Moreover, the definitive and physically not motivated assumption of the Riemannean curvature as a

mathematical adequate of the gravitational field in GR does not contribute to a full comprehension of why the computations in the theory meet the experiment in the Solar system (and even out of it) so well. This circumstance, being so charming in the early days of the theory, may generate some hesitations, because 80 years seem to be long enough time for the clarification of this fundamental for GR problem. Together with the well known "energy-momentum problems", this may lower the authority of GR.

In gauge theories the curvature, considered as generated by connections on principle bundles for some groups and their representations, leading to linear connections in vector bundles, is also a *leading concept*. For example, the energy-momentum tensor in these theories is a quadratic expression of the curvature, and the significance of energy and momentum in modern classical and quantum gauge theories and in the whole physics at all is out of any doubt. Except electrodynamics, where we have much enough experience, in other gauge theories there is also no enough motivation for using namely the curvature as a mathematical adequate of a physical field. The consideration of connections as basic mathematical objects in physical theories we do not consider as sufficiently legalizing move for the introduction of curvatures, although the connections define derivative laws for the sections of vector bundles. In our opinion, a more basic analysis of the question why the curvature works well in physical theories from the point of view of the Frobenius integrability theory would contribute to a more complete and detailed understanding of this important moment in the field theories.

In mathematics the curvature defines those conditions, at which, given differential relations determine integral objects, or what are the obstacles for building these integral objects. In physics, from Newton's time on, a basic quantity is the *force*, i.e. the quantity of transferred energy-momentum from the outer field to the object under observation (usually test particle(s)) in an unit 4-volume. From the point of view of the outer field this means *loss* of energy-momentum, and that's why it does not seem reasonable to expect a Frobenius integrability for the equations, describing the outer field, if these equations do not take into account these losses. It is quite illogical and unreasonable to expect that the expressed through differential relationships properties of a physical object should define it entirely (or integrally) if there is some "flowing out" of substance towards other physical objects, if this "flowing out" takes some energy-momentum from it and carries it to the other objects. In other words, the interaction, if it is not fully described, may violate the existing for any extended object connection between its differential

(local) and integral properties. And a fully described interaction means to say, and to take into account mathematically, where the energy-momentum losses go to. These losses are just the *force*, applied onto the other object(s).

The curvature of a given differential system $\mathcal{P}$ measures mathematically (at a given point and in a differential way) something very close to a "flowing out" of $\mathcal{P}$, namely, it determines *what parts of the Lie brackets of vector fields in* $\mathcal{P}$ *belong no more to* $\mathcal{P}$. We may say, that these parts of the Lie brackets, namely the vertical projections of the Lie brackets, define local flows, directed out of $\mathcal{P}$. Therefore, from physical point of view it is natural to choose curvatures as measuring quantities of the external exertion on test particles in outer fields. The physical measure of this exertion is a change in the *universally conserved quantities energy and momentum* (of the particle).

As far as test (point-like) particles in outer fields are considered the usage of curvature as a direct participant in describing the energy-momentum transfer seems natural, since we are not interested in the dynamics of the external fields. "No change in the energy-momentum of the particle" is equal to "no presence of external fields", i.e. the force is zero. This approach works no more in case of local interaction of two (or more) continuous objects, i.e. two fields, where together with the local energy-momentum transfer we are interested also in the dynamics of the two interacting fields. So, the local dynamical changes of any of the two interacting fields should depend on 2 things: the proper dynamical character of any of the fields and the kind of interaction. These two components of the physical system should be consistent, and when there is no interaction, the mathematical expression for zero energy-momentum transfer should also be consistent with the proper dynamics of any of the fields, considered now as free. Because of the local character of the interaction, derivatives of the fields' components participate in the corresponding mathematical expression. In this way, putting the interaction expression equal to zero, we obtain general enough differential equations for the free fields. And if the fields are mathematically described by curvatures we obtain differential equations for the curvatures.

Let's consider now a more complicated situation, when the interaction energy-momentum flows along *several* channels. Every such channel may or may not produce dissipation. The produced by some of the channels dissipation of energy-momentum could be utilized or not by some of the rest channels. Our proposal to interpret mathematically the non-zero dissipation by Frobenius non-integrability would imply in such cases, that some of the subdistributions of the initial distribution may be nonintegrable, but these

nonintegrable subdistributions may participate in integrable subdistributions of a higher dimension. The same can be said about the corresponding Pfaff systems. This establishes some kind of hierarchy among the subsystems of the Pfaff systems and leads to a more complete study of the initially assumed Pfaff system, defining the energy-momentum interaction channels.

We shall see in the next section that the considered examples-solutions of our equations, representing a (3+1)-interpretation of all known (1+1)-soliton solutions illustrate the above outlined idea: availability of differential and integral conservation laws, non-integrability of 1-dimensional Pfaff subsystem, integrability of all 2-dimensional Pfaff systems. This is in the general spirit of our approach, in which the leading ideas are the extended character of the real objects and the interrelation between their local and integral properties.

We are not going to consider here non-utilized dissipations at the highest possible level. This would lead us far behind our purposes.

After these mathematical and interpretational deviations we go back to EED. First we note that our base manifold, where all fields and operations are defined, is the simple 4-dimensional Minkowski space. According to our equations (1.43) the medium reacts to the field Ω by means of the two $\mathcal{R}^2$-valued 1-forms Φ and Ψ. So, we obtain four $\mathcal{R}$-valued 1-forms $\alpha^1, \alpha^2, \alpha^3, \alpha^4$. Because of the 4-dimensions of Minkowski space and the kind of the equations (3.9) it is easily seen that only 1-dimensional and 2-dimensional Pfaff systems may be of interest from the Frobenius integrability point of view. All Pfaff systems of higher dimension are trivially integrable. Note also that closed 1-forms and linearly dependent 1-forms define always integrable Pfaff systems in an obvious way.

The integrability equations for 1-dimensional Pfaff systems are

$$\mathbf{d}\alpha^i \wedge \alpha^i = 0,\ i = 1, 2, 3, 4. \tag{3.14}$$

Every of the 4 equations (3.14) is equivalent to 4 scalar nonlinear equations for the components of the corresponding 1-form. We note, that the solutions of (3.14), as well as the solutions of the general integrability equations for a p-dimensional Pfaff system are determined up to a scalar multiplier, i.e. if α^i are solutions, then $f_i.\alpha^i$ (no summation over i), where f_i are smooth functions, are also solutions.

In case of 2-dimensional Pfaff systems (α^i, α^j), defined by four 1-forms, their maximal number is 4.3=12. The Frobenius equations read

$$\mathbf{d}\alpha^i \wedge \alpha^i \wedge \alpha^j = 0,\ i \neq j. \tag{3.15}$$

We have here 12 nonlinear equations for the 16 components of α^i. Clearly, these equations (3.15) make some interest only if the corresponding α^i, the exteriour differential $\mathbf{d}\alpha^i$ of which participates in (3.15), does not satisfy (3.14).

After these general remarks we pass to finding explicit solutions of (1.43) with non-zero currents.

3.3 Explicit Solutions with Nonzero Currents

3.3.1 Choice of the anzatz and finding the solutions

From purely formal point of view finding of a solution, whatever it is, legitimizes the equations (1.43) and (3.15) as a consistent system. Our purpose, however, is not purely formal, we consider it as physically meaningful, namely, we are interested in solutions, which are *physically interpretable* as models of real objects in the above commented sense. That's why we have to meet the following. First, the solutions must be *physically clear*, which means that the anzatz assumed should be comparatively simple and its choice should be made on the base of a preliminary analysis of the physical situation in view of the mathematical model used. Second, *it is absolutely obligatory the solutions to have well defined integral energy and momentum*. Third, *to be in the spirit of the soliton-like comprehension of the real natural objects* when the solution is interpreted as a model of such an object. Fourth, to realize the above given *physical interpretation of the Frobenius integrability equations* and the mentioned *dimensional hierarchy of Pfaff systems as a possible model of suitably chosen and intrinsically structured interrelated processes.* Finally, it would be nice, the solutions found to be comparatively simple and interesting as corresponding generalizations, or extensions, of "popular" and well known solutions of "well liked" equations. Probably not all solutions will satisfy these requirements, but there must be such solutions, since the physical significance of our equations depends strongly on this circumstance, to admit solutions with the above mentioned features. Let's now get started.

The first, we take into account, is the necessary time-dependence of the solutions, therefore, the "electric" and the "magnetic" components should present. The simplest Ω, or $(F, *F)$, meeting this requirement, looks as follows (we use the above assumed notations):

$$F = -u dy \wedge dz - v dy \wedge d\xi, \ *F = v dx \wedge dz + u dx \wedge d\xi. \tag{3.16}$$

When choosing the 1-forms α^i we shall obey the requirement, that the "medium" *does not involve* in the $F \leftrightarrow *F$ exchange, so we put

$$\alpha^2 = \alpha^3 = 0. \tag{3.17}$$

The rest two 1-forms , α^1 and α^4 must be *linearly independent*. This would be guaranteed if we choose one of them to be *time-like*, and the other to be *space-like*. The simplest time-like 1-form is of course $\alpha = A(x,y,z,\xi)d\xi$, moreover, any such 1-form defines *integrable* 1-dimensional Pfaff system:

$$\mathbf{d}\alpha \wedge \alpha = (A_x dx \wedge d\xi + A_y dy \wedge d\xi + A_z dz \wedge d\xi) \wedge A d\xi = 0.$$

The choice of the last one, denoted by β, reads $\beta = \beta_1 dx + \beta_2 dy + \beta_3 dz$. Clearly, $\beta^2 < 0$, and $\alpha.\beta = 0$, so they are independent. Now, β participates in the equations through the expression $\beta \wedge *F$, and from the explicit form of $*F$ is seen, that the coefficient β_1 in front of dx does not take part in the equations, therefore, we put $\beta_1 = 0$. So, we get

$$\alpha^1 \equiv \beta = b dy - B dz,\ \ \alpha^4 \equiv \alpha = A d\xi,\ \ \alpha^2 = \alpha^3 = 0. \tag{3.18}$$

We note, that the so chosen β does *not* define in general an integrable 1-dimensional Pfaff system, so the requirement for *hierarchy* of the exchange processes, i.e. $\mathbf{d}\alpha \wedge \alpha = 0,\ \mathbf{d}\beta \wedge \beta \neq 0,\ \mathbf{d}\beta \wedge \beta \wedge \alpha = 0$, is obeyed at a definite level.

At these conditions our equations

$$\delta * F \wedge F = \alpha \wedge F,\ \delta F \wedge *F = \beta \wedge *F,\ \delta * F \wedge *F - \delta F \wedge F = 0,$$

$$\mathbf{d}\alpha \wedge \alpha \wedge \beta = 0,\ \mathbf{d}\beta \wedge \alpha \wedge \beta = 0$$

take the form: $\delta * F \wedge *F - \delta F \wedge F = 0$ is reduced to

$$-v u_y + u v_y = 0,\ -u v_x + v u_x = 0,$$

the Frobenius equations $\mathbf{d}\alpha \wedge \alpha \wedge \beta = 0,\ \mathbf{d}\beta \wedge \alpha \wedge \beta = 0$ reduce to

$$(-b_x B + B_x b)\,.A = 0,$$

$\delta * F \wedge F = \alpha \wedge F$ is reduced to

$$u\,(u_\xi - v_z) = 0,\ v\,(u_\xi - v_z) = 0,\ u u_x - v v_x = A u,$$

$\delta F\wedge *F=\beta\wedge *F$ reduces to

$$v\left(v_\xi-u_z\right)=-bv,\; u\left(v_\xi-u_z\right)=-bu,\; uu_y-vv_y=Bu.$$

In this way we obtain 7 equations for 5 unknown functions u,v,A,B,b. The equations $-vu_y+uv_y=0,\; -uv_x+vu_x=0$ have the following solution:

$$u(x,y,z,\xi)=f(x,y)U(z,\xi),\; v(x,y,z,\xi)=f(x,y)V(z,\xi).$$

That's why

$$AU=f_x\left(U^2-V^2\right),\; BU=f_y\left(U^2-V^2\right),\; f.\left(U_z-V_\xi\right)=b,\; U_\xi-V_z=0.$$

Now the equation $B_xb-Bb_x=0$ takes the form

$$ff_{xy}=f_xf_y,$$

and the general solution of this equation is $f(x,y)=g(x)h(y)$. Besides, the equation $gh\left(V_\xi-U_z\right)=-b$ requires $b(x,y,z,\xi)=g(x)h(y)b^o(z,\xi)$, so we get

$$V_\xi-U_z=-b^o.$$

The relations obtained show how to build a solution of this class. Namely, first, we choose the function $V(z,\xi)$, then we determine the function $U(z,\xi)$ by

$$U(z,\xi)=\int V_z d\xi+l(z),$$

where $l(z)$ is an arbitrary function, which may be assumed equal to 0. After that we define $b^o=U_z-U_\xi$. The functions $g(x)$ and $h(y)$ are arbitrary, and for A and B we find

$$A(x,y,z,\xi)=g'(x)h(y)\frac{U^2-V^2}{U},\quad B(x,y,z,\xi)=g(x)h'(y)\frac{U^2-V^2}{U}.$$

In this way we obtain a family of solutions, which is parametrized by one function V of the two variables (z,ξ) and two functions $g(x),\; h(y)$, each depending on one variable.

For $*(\alpha\wedge F+\beta\wedge *F)$ we obtain

$$*(\alpha\wedge F+\beta\wedge *F)=Audx-Budy-budz-bvd\xi=$$

$$=\frac{1}{2}(U^2-V^2)\left[(gh)^2\right]_x dx-\frac{1}{2}(U^2-V^2)\left[(gh)^2\right]_y dy-$$

$$-(gh)^2\left(\int Ub^o dz\right)_z dz-(gh)^2\left(\int Vb^o d\xi\right)_\xi d\xi == -\left\{\frac{\partial}{\partial x^\nu}P^\nu_\mu\right\}dx^\nu,$$

where the interaction energy-momentum tensor is defined by the matrix

$$P^\nu_\mu=\left\|\begin{array}{cccc} -\frac{1}{2}(gh)^2 Z & 0 & 0 & 0\\ 0 & \frac{1}{2}(gh)^2 Z & 0 & 0\\ 0 & 0 & (gh)^2\int Ub^o dz & 0\\ 0 & 0 & 0 & (gh)^2\int Vb^o d\xi \end{array}\right\|,$$

and the notation $Z\equiv U^2-V^2$ is used. For the full energy tensor $T^\nu_\mu=Q^\nu_\mu+P^\nu_\mu$ we obtain

$$T^3_3=(gh)^2\left[\int Ub^o dz-\frac{1}{2}(U^2+V^2)\right],$$

$$T^4_3=-T^3_4=(gh)^2 UV,$$

$$T^4_4=(gh)^2\left[\int Vb^o d\xi+\frac{1}{2}(U^2+V^2)\right],$$

and all other components are zero.

3.3.2 Examples

In this subsection we consider some of the well known and well studied (1+1)-dimensional soliton equations as generating procedures for choosing the function $V(z,\xi)$, and only the 1-soliton solutions will be explicitly elaborated. Of course, there is no anything standing in our way to consider other (e.g. multisoliton) solutions. We do not give the corresponding formulas just for the sake of simplicity.

We turn to the soliton equations mainly because of two reasons. First, most of the solutions have a clear physical sense in a definite part of physics and, according to our opinion, they are sufficiently attractive for models of real physical objects with internal structure. Second, the soliton solutions describe free and interacting objects with *no dissipation* of energy and momentum, which corresponds to our interpretation of the Frobenius integrability equations, as we explained in the preceding (sub)sections.

1. Nonlinear equation Klein-Gordon. In this example we define our functions U and V through the derivatives of the function $f(z,\xi)$ in the following way:

$U = f_z,\ V = f_\xi$. Then the equation $U_\xi - V_z = f_{z,\xi} - f_{\xi z} = 0$ is satisfied automatically, and the equation $U_z - V_\xi = b^o$ takes the form $f_{zz} - f_{\xi\xi} = b^o$. Since b^o is unknown, we may assume $b^o = b^o(f)$, which reduces the whole problem to solving the general nonlinear Klein-Gordon equation when b^o depends nonlinearly on f. Since in this case $V = f_\xi$ we have

$$\int V b^o(f) d\xi = \int f_\xi b^o(f) d\xi = \int \left[\frac{\partial}{\partial\xi}\int b^o(f)df\right] d\xi = \int b^o(f)df.$$

For the full energy density we get

$$T_4^4 = \frac{1}{2}(gh)^2\left\{f_z^2 + f_\xi^2 + 2\int b^o(f)df\right\}.$$

Choosing $b^o(f) = m^2 sin(f)$ we get the well known and widely used in physics Sine-Gordon equation, and together with this, we can use *all* solutions of this nonlinear equation. When we consider the (3+1) extension of the soliton solutions of this equation, the functions $g(x)$ and $h(y)$ have to bo localized too. The determination of the all 5 functions in our approach is straightforward, so we obtain a (3+1)-dimensional version of the soliton solution chosen. As it is seen from the above given formula, the integral energy of the solution differs from the energy of the corresponding (1+1)-dimensional solution just by the (x,y)-localizing factor $(gh)^2$.

For the 1-soliton solution (kink) we have:

$$f(z,\xi) = 4arctg\left\{exp\left[\pm\frac{m}{\gamma}(z - \frac{w}{c}\xi)\right]\right\},\ \gamma = \sqrt{1 - \frac{w^2}{c^2}}$$

$$U(z,\xi) = f_z = \frac{\pm 2m}{\gamma ch\left[\pm\frac{m}{\gamma}\left(z - \frac{w}{c}\xi\right)\right]},\ V(z,\xi) = f_\xi = \frac{\pm 2mw}{c\gamma ch\left[\pm\frac{m}{\gamma}\left(z - \frac{w}{c}\xi\right)\right]},$$

$$A = g'(x)h(y)\frac{\pm 2m\gamma}{ch\left[\pm\frac{m}{\gamma}\left(z - \frac{w}{c}\xi\right)\right]},\ B = g(x)h'(y)\frac{\pm 2m\gamma}{ch\left[\pm\frac{m}{\gamma}\left(z - \frac{w}{c}\xi\right)\right]},$$

$$b^o = U_z - V_\xi = \frac{-2m^2 sh\left[\pm\frac{m}{\gamma}\left(z - \frac{w}{c}\xi\right)\right]}{ch\left[\pm\frac{m}{\gamma}\left(z - \frac{w}{c}\xi\right)\right]},\ T_4^4 = \frac{(gh)^2 4m^2}{\gamma^2 ch^2\left[\pm\frac{m}{\gamma}\left(z - \frac{w}{c}\xi\right)\right]}$$

and for the 2-form F we get

$$F = -\frac{\pm 2mg(x)h(y)}{\gamma ch\left[\pm\frac{m}{\gamma}(z - \frac{w}{c}\xi)\right]}dy\wedge dz + \frac{w}{c}\frac{\pm 2mg(x)h(y)}{\gamma ch\left[\pm\frac{m}{\gamma}(z - \frac{w}{c}\xi)\right]}dy\wedge d\xi.$$

In its own frame of reference this soliton looks like

$$F=-\frac{\pm 2mg(x)h(y)}{ch(\pm mz)}dy\wedge dz.$$

From this last expression and from symmetry considerations, i.e. at homogeneous and isotropic medium, we come to the most natural choice of the functions $g(x)$ and $h(y)$:

$$g(x)=\frac{1}{ch(mx)},\quad h(y)=\frac{1}{ch(my)}.$$

2. *Korteweg-de Vries equation.* This nonlinear equation has the following general form:

$$f_\xi+a_1ff_z+a_2f_{zzz}=0,$$

where a_1 and a_2 are 2 constants. The well known 1-soliton solution is

$$f(z,\xi)=\frac{a_o}{ch^2\left[\frac{z}{L}-\frac{w}{cL}\xi\right]},\ L=2\sqrt{\frac{3a_2}{a_oa_1}},\ w=\frac{ca_oa_1}{3}.$$

We choose $V(z,\xi)=f(z,\xi)$ and get

$$U=-\frac{a_oc}{w}\frac{1}{ch^2\left[\frac{z}{L}-\frac{w}{cL}\xi\right]},\ b^o=U_z-V_\xi=\left(\frac{c}{Lw}-\frac{w}{c}\right)\frac{2a_o}{ch^3\left[\frac{z}{L}-\frac{w}{cL}\xi\right]},$$

$$T_4^4=(gh)^2\frac{a_o^2c^2(1+L)}{2w^2Lch^4\left[\frac{z}{L}-\frac{w}{cL}\xi\right]}.$$

3. *Nonlinear Schroedinger equation.* In this case we have an equation for a complex-valued function, i.e. for two real valued functions. The equation reads

$$if_\xi+f_{zz}+|f|^2f=0,$$

and its 1-soliton solution, having oscillatory character, is

$$f(z,\xi)=2\beta^2\frac{exp\left[-i\left(2\alpha z+4(\alpha^2-\beta^2)\xi-\theta\right)\right]}{ch\left(2\beta z+8\alpha\beta\xi-\delta\right)}.$$

The natural substitution $f(z,\xi) = \sqrt{\rho}.exp(i\varphi)$ brings this equation to the following two equations

$$\rho_\xi + (2\rho\varphi_z)_z = 0,\ 4\rho + \frac{2\rho\rho_{zz} - \rho_z^2}{\rho^2} = 4(\varphi_\xi + \varphi_z^2).$$

For the 1-soliton solution we get

$$\rho = \frac{4\beta^4}{ch^2\left(2\beta z + 8\alpha\beta\xi - \delta\right)},\ \varphi = -\left[2\alpha z + 4(\alpha^2 - \beta^2)\xi - \theta\right].$$

We put $U = \rho,\ V = -2\rho\varphi_z$ and obtain

$$U = \rho = \frac{4\beta^4}{ch^2\left(2\beta z + 8\alpha\beta\xi - \delta\right)},\ V = -2\rho\varphi_z = \frac{16\alpha\beta^4}{ch^2\left(2\beta z + 8\alpha\beta\xi - \delta\right)},$$

$$A = g'(x)h(y)\frac{4\beta^4(1 - 16\alpha^2)}{ch^2\left(2\beta z + 8\alpha\beta\xi - \delta\right)},\ B = g(x)h'(y)\frac{4\beta^4(1 - 16\alpha^2)}{ch^2\left(2\beta z + 8\alpha\beta\xi - \delta\right)},$$

$$b^o = \frac{16\beta^3(16\alpha^2\beta^2 - 1)sh(2\beta z + 8\alpha\beta\xi - \delta)}{ch^3(2\beta z + 8\alpha\beta\xi - \delta)},$$

$$T_4^4 = (gh)^2\frac{16\beta^8}{ch^4(2\beta z + 8\alpha\beta\xi - \delta)}.$$

We note that the solution of our equations obtained has no the oscillatory character of the original Schroedinger 1-soliton solution, it is a (3+1)-localized running wave and moves as a whole with the velocity $4c\alpha$.

4.*Boomerons.* The system of differential equations, having soliton solutions, known as *boomerons*, is defined by the following functions: $\mathbf{K} : \mathcal{R}^2 \to \mathcal{R}^3,\ H : \mathcal{R}^2 \to \mathcal{R}$, and besides, two constant 3-dimensional vectors $\mathbf{r}$ and $\mathbf{s}$, where $\int$ is a unit vector: $|\mathbf{s}| = 1$. The equations have the form

$$H_\xi - \mathbf{s}.\mathbf{K}_z = 0,\ \mathbf{K}_{z\xi} = H_{zz}\mathbf{s} + \mathbf{r} \times \mathbf{K}_z - 2\mathbf{K}_z \times (\mathbf{K} \times \mathbf{s}).$$

Now we have to define our functions $U(z,\xi),\ V(z,\xi)$ and $b^o(z,\xi)$. The defining relations are:

$$U = H_z = |\mathbf{s}|^2 H_z,\ V = \mathbf{s}.\mathbf{K}_z = (\mathbf{s}.\mathbf{K})_z,\ b^o = -\mathbf{s}.\left[\mathbf{r} \times \mathbf{K}_z - 2\mathbf{K}_z \times (\mathbf{K} \times \mathbf{s})\right].$$

Under these definitions our equations $U_\xi - V_z = 0,\ V_\xi - U_z = -b^o$ look as follows:

$$[H_z - \mathbf{s}.\mathbf{K}_z]_z = 0,\ \mathbf{s}.\left[\mathbf{K}_{z\xi} - H_{zz}\mathbf{s} - \mathbf{r}\times\mathbf{K}_z + 2\mathbf{K}_z\times(\mathbf{K}\times\mathbf{s})\right] = 0.$$

It is clear that every solution of the "boomeron" system determines a solution of our system of equations according to the above given rules and with the multiplicative factors $g(x)$ and $h(y)$. The two our functions $A(z,\xi)$ and $B(z,\xi)$ are easily then computed.

Following this procedure we can generate a solution to our system of equations by means of every solution to any (1+1)-soliton equation, as well to compute the corresponding conserved quantities. It seems senseless to give here these easily obtainable results. The richness of this comparatively simple family of solutions, as well as the availability of corresponding correctly defined integral conserved quantities, are obvious and should not be neglected. In particular, it would be interesting to analyze the abilities of the *breather*-solutions of some soliton equations as possible models of bounded systems of the type of *hydrogen atom*. It is worth to note that in this approach the classical quantity *potential* is out of need. The proton and the electron participate equally in rights as relatively isolated subsystems of a more general dynamical system. The discrete character of the energy spectra trivially follows from the fact, that the transitions among the various stationary states of the more general system are caused by creation or annihilation of photons, taking into the system or out of it the corresponding conserved quantities. As for the quantitative description of this spectra, probably, it will be achieved by a corresponding choice of the integrability constants.

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The author does not know about any other published or not published studies and results following the same or similar line of nonlinear extension of Maxwell's equations. The author would be kindly grateful for any indication of results, having something in common with those given in this review. Any comments and remarks, please, send to

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